The Consumer Credit Channel of Monetary Policy*

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Abstract

Using the Consumer Expenditure Survey, I document a new fact that the consumption response to monetary policy shocks is greater for households with higher default risk. I propose a consumer credit channel that accommodative monetary policy extends credit disproportionately to risky households which have higher propensities to spend out of extended credit. I study the mechanism in a Heterogeneous Agent New Keynesian model augmented with asymmetric information. In the model, credit limits arise because borrowers can default on loans and borrowing signals a risky type. Accommodative monetary policy extends credit as it lowers default rate and changes lenders' beliefs on the types of borrowers. Calibrated to match the cross-sectional distribution of default rate, credit limit, and marginal propensity to spend, the consumer credit channel accounts for 63% of the heterogeneous consumption responses and 20% of the aggregate response. The model is used to assess the distributional effects of monetary policy and the "risk-taking" channel.

Keywords: consumer credit, monetary policy, default risk, adverse selection, heterogeneous agents.

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1 Introduction

A satisfactory understanding of the monetary transmission mechanism is the basis of any effective conduct of monetary policy. The current paper focuses on the transmission mechanism through the largest component of GDP: household consumption. The traditional New Keynesian model emphasizes the *demand* response to changes in the policy rate from a representative consumer, which are inconsistent with two basic facts. On the one hand, macro evidence suggests that the consumption sensitivity to changes in the interest rate is small (Campbell and Mankiw (1989)). On the other hand, micro evidence shows that there exists substantial heterogeneity in borrowing limits, and households facing tight borrowing limits respond strongly to the changes in *quantity* but not the *price* of available credit (Gross and Souleles (2002)). This points to a gap in the literature that the response of credit supply to different households may be important in driving consumption response but is absent in traditional models.

The current paper fills in this gap. Using the Consumer Expenditure Survey, I first document a new fact that the consumption response to monetary policy shocks is greater for households with higher default risk. The results can be best summarized in Figure 1[1]. After a negative innovation in the federal funds rate, the consumption impulse response for households with higher default risk is two times larger than the average, while it is virtually zero for those with lower default risk. I study a model featuring endogenous credit extensions in both prices and quantities that explains the heterogenous consumption responses. I use the model to answer the question: what is the role of consumer credit extensions in driving heterogeneous and aggregate consumption responses to monetary policy shocks?

The model builds on the Bewley-Huggett-Aiyagari incomplete-market heterogeneous-agent framework, incorporating information asymmetry and a standard New Keynesian block. In the model, infinitely lived households receive idiosyncratic labor efficiency shocks, value leisure and consumption, and save and borrow by trading discount bonds with competitive financial intermediaries. Households can default on their loans. Financial intermediaries factor the default risk into the bond price, and endogenous credit limits arise.

The core of the model mechanism lies in that credit is rationed due to adverse selection, which is alleviated by accommodative monetary policy. Households differ in their default risk since they discount future default cost differently. But risk types are private information. Financial intermediaries thus cannot condition bond prices on households'

[1] Detailed empirical analysis is in Section 2.

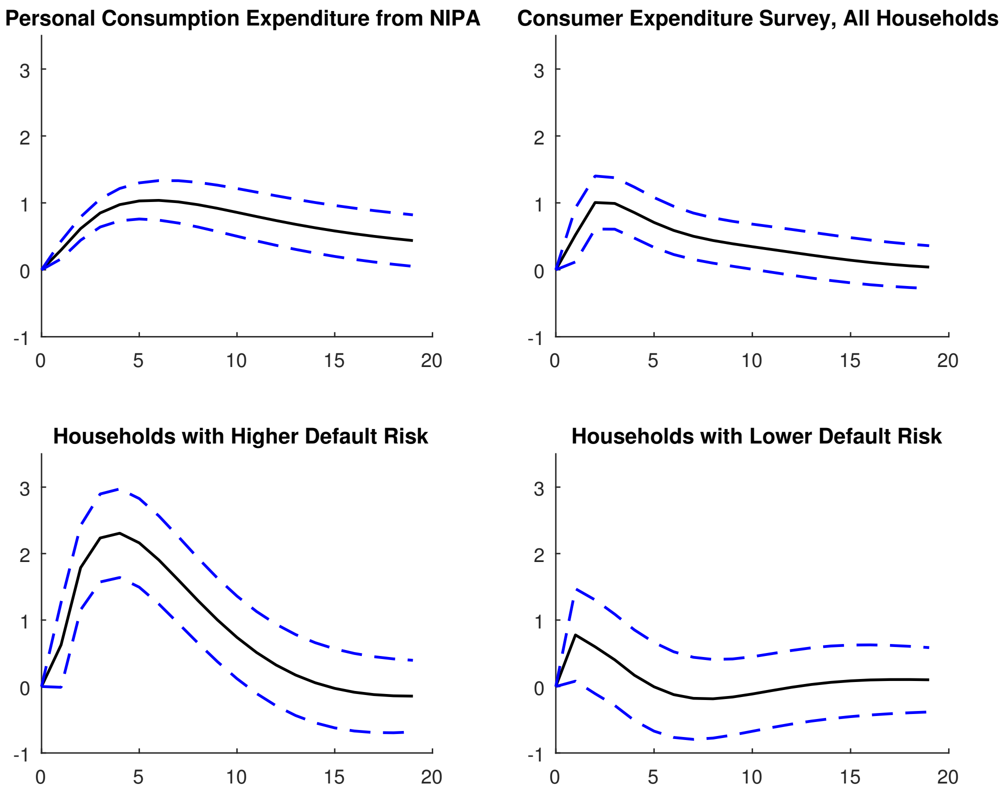

Figure 1: Impulse Response of Consumption

Notes: Units on the horizontal axis are quarters. Units on the vertical axis are percentage points. The solid lines are the impulse responses to a 100 basis point negative innovation in federal funds rate. The dashed lines are one standard deviation intervals constructed by bootstraps with 200 repetitions. Households' default risk is measured based on risk premium charged on their auto loan interest rates. Monetary policy shocks are identified by ordering the federal funds rate last in a SVAR. The data in use is from 1984Q1 to 2007Q4.

risk types but can make inferences from their choices. In normal times when the interest rate is high, the risky type borrows more than the safe type since the former is less patient, borrowing thus signals a risky type, and credit is rationed. A temporary cut in the real interest rate encourages the safe type to increase borrowing *relatively more* than the risky type. The posterior probability of being a risky type conditional on borrowing decreases and credit is extended.

To motivate an ex-ante measure of households' default risk and associate the model with data, I assume financial intermediaries track a "credit score" of each household similar to Chatterjee et al. (2011), which denotes the prior probability the household is a safe type. I introduce unobservable preference shocks so that types cannot be revealed immediately in a single period. The credit score is thus used to form the expectation of default probability and to price bonds. The credit score is updated following Bayes rule over time and reveals households' types gradually.

There are two important empirical regulations that I use to put quantitative discipline on the model. The first is the salient fact that consumers with higher default risk are also those with higher marginal propensities to spend out of extended credit. The correlation between credit limit and marginal propensity to spend is crucial in determining how much of the extended credit is transformed into final aggregate demand. The second is the extent to which credit limit varies with credit score. This is indicative of the degree of adverse selection in the consumer credit market: if adverse selection is light and consumers' behaviors perfectly reveal their types, prior information (credit score) should not be important in determining credit price and credit limit, and vice versa. I calibrate the model to match the cross-sectional distribution of default rate, credit limit, and marginal propensity to spend in the data.

The main quantitative exercise is to study the transition path of the economy after monetary policy shocks modeled as unexpected shocks to the Taylor rule. I show that the model generates heterogeneous consumption responses qualitatively consistent with data. After a shock that lowers the nominal interest rate by 25 basis point on impact, the consumption response for the lower credit score group is 36% larger than the higher credit score group measured by percentage deviation on impact, or 63% larger measured by cumulated response through the transition.

Heterogeneous consumption responses arise in the model because households differ in their marginal propensities to spend out of extended credit and because they face different responses of credit supply following the monetary policy shock. Credit supply responds to the monetary policy shock for two reasons. First, given the type, the lower risk-free interest rate lowers the cost of rolling over debt and lowers the default rate. Second, the lower borrowing cost encourages the safe type to increase borrowing *relatively* more than the risky type, makes the pool of borrowers on average safer, and alters lenders' beliefs on the risk types of borrowers.

I examine the quantitative importance of different model mechanisms in explaining consumption responses through a series of counter-factual experiments. I show that the changes in lenders' beliefs account for the majority of heterogeneity in consumption and credit supply responses. If the financial intermediaries were to ignore the changes in borrowing behaviors when making type inferences, the difference in consumption responses would be 31% lower measured by percentage deviation on impact, or 63% lower measured by cumulated response through the transition, and the difference in credit limit responses would virtually disappear.

The changes in lenders' beliefs are also quantitatively important in driving aggregate consumption response. With the responses in lenders' beliefs turned off, the aggregate consumption response is 20% lower on impact. I show that the effect of changes in credit supply on aggregate consumption is of similar magnitude as the effect of changes in the risk-free interest rate, and the latter is the key force in generating consumption response in traditional New-Keynesian models.

The model makes sharp predictions on the distributional effects of monetary policy. While households on average benefit from the accommodative monetary policy shock due to alleviated monopolistic inefficiencies, the consumption-equivalent welfare gain is three times larger for households in the bottom wealth quintile and lower credit score group, than in the top wealth quintile and higher credit score group.

The model speaks to a "risk-taking" channel of monetary policy in three senses: an expansionary monetary policy shock reduces loan risk premium disproportionately for more risky households, channels a larger fraction of aggregate credit to more risky households, and triggers a spike in the aggregate default rate afterwards. I show that it is precisely the changes in lenders' beliefs that lead to the risk-taking channel.

Related Literature. This paper is mainly related to three strands of the literature. The first is a long-standing literature on the credit channel of monetary policy. The current paper differentiates from existing theoretical channels along two important dimensions. First, the traditional credit channel model a la Kiyotaki and Moore (1997) and Bernanke et al. (1999) focuses on the production side, and accommodative monetary policy eases credit by increasing the net worth of entrepreneurs/firms and boosts output by moving capital to the productive sector or increasing investment demand[2]. In the current paper, monetary policy eases credit to risky consumers who have higher propensities to spend and boosts output by increasing consumption demand[3]. Second, classical literature models financial frictions arising from limited enforcement. The current paper is, to my best knowledge, the first to incorporate adverse selection into a DSGE framework with monetary policy[4].

On the empirical side, the evidence on the credit channel is somewhat mixed. While it is a well-established fact that accommodative monetary policy channels credit dispro-

[2]Similarly, in the variant model of Gertler and Karadi (2011) with a banking sector, the positive feedback of credit channel works by increasing net worth of bank's equity, decreasing rate premium, and stimulating investment demand.

[3]In the variant model of Iacoviello (2005) where entrepreneurs choose to both consume and invest housing upon an increase in net worth due to increase in housing price, the increase in aggregate demand comes from the increase in both investment and consumption demand.

[4]Kurlat (2013) and Bigio (2015) are recent developments on incorporating asymmetric information in the study of credit channel. The current paper departs from them by explicitly studying the credit rationing due to adverse selection and assessing the effect of monetary policy.

portionately to small and risky *firms* (Gertler and Gilchrist (1994), Jiménez et al. (2012), Ioannidou et al. (2015), Dell'Ariccia et al. (2016)), the recent paper Agarwal et al. (2015b) uses credit card utilization data and argues that monetary policy failed to extend credit to risky *consumers* during the last financial crisis. Using a longer horizon and times-series based identification method, I find the opposite results to Agarwal et al. (2015b). While their conclusions are based on partial-equilibrium imputations of banks' lending cost using variations across borrowers at a given time, I show that it is crucial in accounting for the general-equilibrium change in the risk composition of borrowers to understand the credit supply responses.

The second related literature is on monetary policy in incomplete-market heterogeneous-agent models. Papers that focus on the transmission mechanism include Auclert (2014), Kaplan et al. (2015), Luetticke (2015), Wong (2015), and McKay et al. (2015). Papers that focus on distributional effects include Gornemann et al. (2014) and Doepke et al. (2015). The current paper is the first one to study a transmission mechanism through the endogenous credit supply response in such a framework. The current paper also offers a new angle through which monetary policy has heterogeneous and distributional effects. On the empirical side, the current paper is closely related to Wong (2015) and Cloyne et al. (2015) who document heterogeneous impulse responses using micro survey data.

The third related literature is on endogenous credit limit and its interactions with public policy. The current paper bridges the Eaton and Gersovitz (1981) style credit limit with the Stiglitz and Weiss (1981) style adverse selection. Therefore it is closed to recent literature on unsecured debt in general equilibrium frameworks and especially when information asymmetry is present. Related papers include Chatterjee et al. (2007), D'Erasmo (2008) (in the context of sovereign default), Chatterjee et al. (2011), and Athreya et al. (2012). The contribution of the current paper along this literature is threefold. First, I show that introducing transitory preference shocks ensures equilibrium existence and provides a natural candidate for off-equilibrium beliefs for unfeasible choices. Second, I derive several analytical properties for how credit limits arise in the model both due to limited enforcement and adverse selection, and how the change in interest rate affects credit rationing. Third, to my best knowledge, this is the first paper to bring such a framework to match cross-sectional facts in the consumer credit market.

The remainder of the paper proceeds as follows. Section 2 describes the empirical analysis. Section 3 describes the model. Section 4 derives several analytical properties of the model. Section 5 takes the model to US data. Section 6 quantitatively studies the transition path after monetary policy shocks. Section 7 concludes.

2 Empirical Analysis

In this section, I show that the consumption impulse response to monetary policy shocks is greater for households with higher default risk. As a summary for the analysis, measures of consumption are constructed using the Consumer Expenditure Survey (CEX) 1984Q1-2007Q4. Households' default risk is measured by risk premia charged on their consumption loans and by imputing propensities of loan delinquency from the Survey of Consumer Finance (SCF). The effects of monetary policy are identified by ordering federal funds rates last in the SVAR and using Romer and Romer (2004) shocks with the local projection method.

2.1 Data and Empirical Strategies

Consumer Expenditure Survey (CEX)

The Consumer Expenditure Survey (CEX) is a quarterly rotating panel continuously collected by the Bureau of Labor Statistics (BLS) since 1980. It consists of an interview part which surveys each households for up to four consecutive quarters. In each interview, households are asked to report their detailed expenditure in different categorizes for the past three months. The survey is designed to represent the whole US population and the interview survey covers the majority of households' expenditures.

Household-level consumption is constructed as the total of durable, nondurable, and service expenditure. Each subcomponent is deflated using the category-specific CPI.

Besides its excellent coverage on expenditure information, CEX also surveys detailed demographics information and the status of durable stocks. In particular, starting from 1984Q1, households are asked to report the principal, remaining balances, and monthly interest payments on their auto loans if they have any, based on which CEX provides imputations for the loan interest rates. Information on when a vehicle was purchased and how it was financed is also provided. This is the key information I use to construct measures of default risk.

Measuring households' default risk

Other things equal, households who are charged with higher interest rates on their auto loans are regarded as having higher default risk. The underlying assumption is that the consumer loans market has priced the default risk into the contracted interest rates, a fact that has been corroborated by Edelberg (2006). I first estimate an OLS equation regressing the auto loan interest rate on a set of observable household and loan characteristics. The regression residual for each observation is extracted as a measure for default risk of a loan. Then I compute the average of the residuals weighted by vehicle values as the measure for default risk for each household since a household may report multiple entries of auto loans. Detailed regression results are reported in Appendix A.2.

For robustness check I also construct an alternative measure of default risk by imputing the propensities of loan delinquency from the Survey of Consumer Finance (SCF).

Monetary policy shocks via R&R

Besides using federal funds rates directly as monetary policy variables, I also use the monetary policy shocks proposed by Romer and Romer (2004) (R&R). The construction first regresses the changes in the target federal funds rate (FFR) after each Federal Open Market Committee (FOMC) meeting on a set of lagged economic performance indicators and the internal forecasts for these indicators[5] from the FOMC Greenbooks. The residuals of the regression are then treated as monetary policy shocks. The idea behind is that the funds rate, which has been the major monetary policy instrument since mid 1980s, is set by the FOMC based on the information gathered for policy making and the information is well represented by the Greenbook forecasts. Therefore, regressing the funds rate on past economic performance indicators removes endogenous policy response to economic conditions; regressing the funds rate on forecasts accounts for policy response arising from expectations. The movement in federal funds rate left unexplained should be treated as policy surprises. I use the series of shocks extended to 2007Q4, right before the era of zero lower bound, by Wieland and Yang (2016).

Identifying the effects of monetary policy shocks using SVAR and local projection

Following Christiano et al. (1999), I adopt a recursive assumption by ordering federal funds rate last in a VAR and construct orthorgonalized impulse responses with the Cholesky decomposition. The identification assumption is that monetary policy is implemented with a lag, and within a period (quarter) economic fundamentals do not respond to monetary policy. I estimate a separate VAR for each group of households. Variables in each VAR include log CEX-measured consumption, log CPI (seasonally adjusted), unemployment rate, log industrial production (seasonally adjusted), and the federal funds rate.

I also complement the analysis using the local projection method as in Jordà (2005)

[5]The particular indicators used in the original paper are inflation, unemployment rate, and output growth.

Coibion et al. (2012), and Ramey (2015). The local projection method regresses the variable of interest in future periods on the current R&R shock and a set of current and lagged controls. Instead of imposing structural assumptions on the dynamics of the system, the effects of the shock on each future period is estimated in a separate equation. Besides working as robustness check, the local projection method is convenient to test the statistical significance of heterogeneous effects.

2.2 Empirical Results

The estimated impulse responses to a 100 basis point negative innovation in the funds rate with SVAR are presented in Figure 1. Lagged control variables are set to 2 periods based on the BIC criteria. Households are split into two groups of equal size based on their ranks of default risk derived from auto loan interest rates. The first panel presents the impulse response of the Personal Consumption Expenditure (PCE) (real, seasonally adjusted) from the NIPA account as a benchmark. The aggregate consumption measured by PCE responds by rising 1% at the peak after 5 quarters, a magnitude in line with the effects on output identified with R&R shocks as in Coibion (2012). The second panel presents the impulse response of average expenditure of all CEX households. The magnitude of the response is similar to the response of PCE. The third and fourth panels present the consumption impulse responses for households with different default risk. The consumption response for the more risky group peaks at 2% and persists to be positive after 20 quarters. The consumption response for the less risky group is smaller on impact, and drops to be insignificant from 0 right after the impact.

The result is robust to using the alternative local projection identification method, as shown in Figure A.2. It is well known impulse responses estimated with the local projection method and R&R shocks are of greater magnitude (Coibion (2012)) and appear more erratic (Ramey (2015)). The current results exhibit similar patterns. The consumption response for the more risky group peaks at 8% and persists after 15 quarters, while the consumption response for the less risky group is insignificant from 0. The differences are statistically significant as reported in Table A.3.

One concern using the grouping method is that households may select in and out of groups in response to the shock. To address this concern, I show that the results are robust based on household-level variations. Detailed analysis is in Appendix A.4.

The result is robust to the alternative measure of default risk imputed from the SCF (Appendix A.5). I also show that the result is not due to differences in other households' characteristics rather than default risk (Appendix A.6). The main goal of the current paper is thus toward a model mechanism that explains the heterogeneous consumption responses.

3 The Model

To capture the dimension of heterogeneity emphasized in the data facts, I model households having different default risk and the difference arises because they discount future default cost differently. Households receive uninsurable idiosyncratic labor shocks, which generates the need for saving and borrowing. Motivated by the fact that lenders have abundant information on consumers' liability status and earnings history, but still rely on external credit rating when offering loan contracts, I assume financial intermediaries cannot observe households' risk types, but can track an ex-ante measure of default risk called "credit score", which reveals types gradually as households make choices over time.

3.1 Households and Market Arrangement

Overall. Time is discrete, indexed by t, starts from 0 and goes to infinity. The economy consists of a continuum of infinitely lived households with constant mass normalized to one and risk-neutral financial intermediaries.

Endowments. At each period t, a household receives an idiosyncratic labor efficiency shock e_t drawn from a finite set $\mathbb{E}$. The shock follows a stationary Markovian process described by transition matrix $\Gamma(e'|e)$. Labor income is earned from labor supply at the hourly wage w_t, and is taxed at flat rate τ_t. Households receive lump sum transfers Tr_t from the government.

Bonds and default choices. Households who do not default on loans can save or borrow from financial intermediaries by transacting one-period non-state contingent discount bonds with face value a' denoted in real dollars[6]. a' takes values from a finite set $\mathbb{A}$ that contains 0. I maintain the convention that positive values of a' denote deposits and negative values denote loans.

Households with loans can choose to declare bankruptcy and discharge all the debt. Households who declare bankruptcy are excluded from the bond market for the current period and cannot save or borrow. Denote d_t as the bankruptcy choice, the set for bond and default choices is thus $\{(d_t, a'_t) : (d_t, a'_t) \in \{0\} \times \mathbb{A} \text{ or } (d_t, a'_t) = (1, 0)\}$.

[6] I assume bond contracts are written in real terms to remove revaluation effects of assets and loans due to inflation/deflation. Doepke et al. (2015) focuses on the distribution effects through this channel.

Preferences. Preferences are additively separable over time. Households first derive flow utility from consumption c_t and dislike labor n_t according to the GHH (Greenwood et al. (1988)) utility function:

$$u_{i_t}(c_t - v(n_t), d_t) = \begin{cases} u(c_t - v(n_t)), \text{ for } d_t = 0, \\ u(\phi_{i_t}[c_t - v(n_t)]), \text{ for } d_t = 1. \end{cases}$$

I assume declaring bankruptcy ($d_t = 1$) incurs utility loss proportional to the consumption-leisure bundle. The utility loss can be interpreted as stigma or any inconvenience associated with carrying a bankruptcy record.

Households can be of either a risky (bad) type or a safe (good) type, denoted by $i_t \in \{b, g\}$. Each type is associated with a different discount factor β_{i_t} with $\beta_g > \beta_b$, and (potentially) different default cost captured by the penalty parameter ϕ_{i_t}. Households face idiosyncratic type-switching shocks with transition matrix Ω.

Households draw transitory preference shocks over actions. Following the discrete choice literature, the preference shock over each feasible choice $\varepsilon_t^{(d_t, a'_t)}$ is additive to the flow utility and drawn i.i.d. from the Type I extreme value distribution with scale parameter σ_ε. Therefore, the total flow utility from choices is equal to

$$u_{i_t}(c_t - v(n_t), d_t) + \varepsilon_t^{(d_t, a'_t)}.$$

Bond price. A bond contract is described by a pair of discount price and value (q, a'). Financial intermediaries are risk-neutral, one-period lived, and operate competitively. Therefore, for each loan level $a' < 0$, financial intermediaries form expectation on the default probability of households that take the loan and set bond price. Competitive pricing indicates that every bond contract just breaks even in equilibrium. Financial intermediaries observe some but not all characteristics of households and therefore the bond price depends on the level a', as well as observable characteristics denoted by x_t. The bond price function at time t is denoted by $q_t(a', x_t)$.

Private information and credit score. I assume financial intermediaries observe labor efficiency shock e_t and current bond holding a_t of households, but not the risk type i_t or the transitory preference shocks. Instead, financial intermediaries make type inferences from households' bond and default choices. Due to the transitory preference shocks, every feasible action is chosen with positive probability by both types, and therefore types cannot be revealed in a single period. Instead, financial intermediaries track a "credit score" s_t of each household, which is the probability that the households is a safe type. Bond and default choices are thus combined with this prior information to form posterior probability that a household is a safe type, based on which financial intermediaries form expectation on the future repay probability and determine bond price. Thus, the bond price function $q_t(a', x_t)$ takes the credit score s_t as argument. Combined with other observable characteristics, x_t is thus the vector (e_t, a_t, s_t).

The updated posterior is carried over time and serves as the prior for the next period. I denote the belief updating or credit scoring function $\psi_t^{(d,a')}(e_t, a_t, s_t)$, which is a function of bond and default choices (d, a'), observable characteristics e_t and a_t, and the current credit score s_t. The credit score reveals types over time, but due to the type-switching shocks, types are never fully revealed and a non-degenerate stationary distribution of credit score emerges over $\mathbb{S}$, a proper subset of $[0, 1]$.

Discussions on the cost of bankruptcy. The cost of bankruptcy in the model first arises from the static utility cost. In addition, if in equilibrium the risky type is more likely to default, the default choice signals a risky type and lowers the future credit score through the credit scoring function ψ_t. Therefore, the cost of bankruptcy also arises from endogenous erosion of *future* reputation. Since the risky type is assumed to have a lower discount factor, he indeed discounts the future cost more and is more likely to default than the safe type. In equilibrium, financial intermediaries thus make consistent inferences by assigning a lower credit score to the bankrupt households. However, it turns out the above force is not strong enough to generate the positive correlation between default rate and credit score close to data. Therefore, in the quantitative evaluation I do allow the static default cost to differ across types.

Discussions on the transitory preference shocks. The introduction of additive transitory preference shocks is unconventional to the standard consumption-saving model. Despite the technical necessity to smooth out the policy function so as to ensure equilibrium existence, the shocks can be viewed as random reputation cost for the bankruptcy decision, or arising from the fact that consumers make mistakes by choosing sub-optimal loan contracts (Agarwal et al. (2015a), Gross and Souleles (2002)).

I now formally describe households' and financial intermediaries' problems.

3.2 Decision Problems for Households

The decision problems for households are formulated recursively. At each period, a household can be characterized by state variables $(i, e, a, s, \boldsymbol{\varepsilon})$, of which $i \in \{b, g\}$ is the risk type, $e \in \mathbb{E}$ is labor efficiency shock, $a \in \mathbb{A}$ is the bond holding, $s \in \mathbb{S}$ is the credit score, and $\boldsymbol{\varepsilon} = \{\varepsilon^{(d,a')}\}$ is the vector of action-specific preference shocks. Denote the observable state as $x = (e, a, s)$.

Denote $\mathbb{Y} = \{(d, a') : (d, a') \in (0 \times \mathbb{A}) \text{ or } (d, a') = (1, 0)\}$ as the bond and default choice set. Denote $\mathbb{M}_t(e, a, s) \subseteq \mathbb{Y}$ the feasible set containing all actions that generate strictly positive consumption leisure bundle, i.e. $\mathbb{M}_t(e, a, s) = \{(d, a') \in \mathbb{Y} : c_t^{(d,a')}(e, a, s) - v(n_t(e)) > 0\}$, where $c_t^{(d,a')}(e, a, s)$ and $n_t(e)$ are consumption and labor decision rules defined later in this subsection. It should be clear that since bond price depends on observable characteristics only, so does the feasible set.

Households take as given wages, proportional labor tax rate, and transfers (w_t, τ_t, Tr_t), the bond price function $q_t(a', e, a, s)$, and the credit scoring function $\psi_t^{(d,a')}(e, a, s)$. The decision problems are described by the Bellman equation:

$$
\begin{aligned}
V_t(i, e, a, s, \boldsymbol{\varepsilon}) = & \max_{(d,a') \in \mathbb{M}_t(e,a,s)} U_t^{(d,a')}(i, e, a, s) + \varepsilon^{(d,a')} \\
& + \beta_i E[V_{t+1}(i', e', a', \psi_t^{(d,a')}(e, a, s), \boldsymbol{\varepsilon}')|i, e],
\end{aligned}
$$

where the expectation operator is w.r.t. the type switching, labor efficiency, and transitory preference shocks. $U_t^{(d,a')}(i, e, a, s)$ is the period return function. For households that do not default,

$$
\begin{aligned}
& U_t^{(0,a')}(i, e, a, s) = \max_{c,n} \; u(c - v(n)) \\
& s.t. \quad c + q_t(a', e, a, s)a' = (1 - \tau_t)w_t e n + a + Tr_t,
\end{aligned}
$$

and for households that default,

$$
\begin{aligned}
& U_t^{(1,0)}(i, e, a, s) = \max_{c,n} \; u(\phi_i[c - v(n)]) \\
& s.t. \quad c = (1 - \tau_t)w_t e n + Tr_t.
\end{aligned}
$$

The policy correspondence to individual households' optimization problem may not be a singleton set everywhere. Following the discrete choice literature (e.g. Rust (1987)), I integrate choices over the transitory preference shocks $\boldsymbol{\varepsilon}$. With the type I extreme value shock, the integrated value and choice functions have explicit aggregation.

Denote $m_t^{(d,a')}(i, e, a, s)$ the probability that a household with state variable (i, e, a, s) chooses action (d, a'). Denote $W_t(i, e, a, s)$ the value integrated over $\boldsymbol{\varepsilon}$:

$$
W_t(i, e, a, s) = E[V_t(i, e, a, s, \boldsymbol{\varepsilon})|i, e, a, s].
$$

Lemma 2 in Appendix B.1 establishes the existence and aggregation results for $m_t(\cdot)$ and $W_t(\cdot)$. I call $m_t(\cdot)$ the inter-temporal policy function and $W_t(\cdot)$ the value function for the rest of the paper.

With the GHH preferences, given current period wage w_t and tax rate τ_t, the policy function for labor supply is a function of the labor efficiency shock only (Lemma 3), denoted by $n_t(e)$. Then the consumption level for each bond and default choice can be written explicitly:

$$c_t^{(d,a')}(e,a,s) = \begin{cases} (1-\tau_t)w_t e n_t(e) + a + Tr_t - q_t(a',e,a,s)a', \text{ for } d=0, \\ (1-\tau_t)w_t e n_t(e) + Tr_t, \text{ for } d=1. \end{cases}$$

3.3 Zero Profit Conditions for Financial Intermediaries

Each financial intermediary takes as given the risk free interest rate r_t, the intertemporal policy function $m_t^{(d,a')}(i,e,a,s)$, and the score updating function $\psi_t^{(d,a')}(e,a,s)$. The bond price as a function of bond value and observable characteristics satisfies the following zero-profit condition:

$$q_t(a',e,a,s) = \frac{\sum_{e'\in\mathbb{E}} \Gamma(e'|e)\ pr_{t+1}(e',a',\psi_t^{(0,a')}(e,a,s))}{1+r_t}, \tag{1}$$

where $\Gamma(e'|e)$ is the transition matrix for labor efficiency shocks. $pr_{t+1}(e,a,s)$ is the probability of repayment defined as:

$$pr_{t+1}(e,a,s) = s(1-m_{t+1}^{(1,0)}(g,e,a,s)) + (1-s)(1-m_{t+1}^{(1,0)}(b,e,a,s)).$$

Recall from the definition of policy function, $(1-m_{t+1}^{(1,0)}(i,e,a,s))$ is the probability of repayment for type i with observable characteristics (e,a,s). By definition, credit score s is the probability of being a safe type. Therefore, $pr_{t+1}(\cdot)$ is the probability of repayment for households with observable characteristics (e,a,s) at period $t+1$.

For every loan contracted at period t, financial intermediaries get full repayment if and only if households repay at period $t+1$. The zero-profit condition dictates that in equilibrium, the expected amount to be repaid next period discounted by the real risk-free interest rate should be equal to the amount received by households in the current period, described by Equation (1).

One important observation is that the credit score (belief) updating function $\psi_t^{(d,a')}(\cdot)$ enters the bond price function. This captures the effects that households signal their types with bond and default choices.

3.4 Credit Score (Belief) Updating Function

Financial intermediaries make inferences on households' types, combining households' choices and the prior information.

The posterior probability that a household is a safe type conditional on observable characteristics x and choices (d, a'), denoted by $\xi_t^{(d,a')}(x)$, is given by the Bayes rule:

$$\xi_t^{(d,a')}(x) = Pr_t(g|x,(d,a')) = \frac{m_t^{(d,a')}(g,x) \cdot s}{m_t^{(d,a')}(g,x) \cdot s + m_t^{(d,a')}(b,x) \cdot (1-s)}. \tag{2}$$

Then after accounting for the type switching shock, the credit score updating function is given by:

$$\psi_t^{(d,a')}(x) = \Omega(g|g)\xi_t^{(d,a')}(x) + \Omega(g|b)[1 - \xi_t^{(d,a')}(x)].$$

3.5 Firms

The New-Keynesian block consists of firms and government as monetary authority. Firms' optimization conditions generate the New-Keynesian Phillips curve linking inflation to marginal cost of production.

Final goods firms. A representative final goods firm uses intermediate goods as inputs and produces according to the standard CES technology:

$$Y_t = (\int_0^1 y_{j,t}^{\frac{\eta-1}{\eta}} dj)^{\frac{\eta}{\eta-1}},$$

where $\eta > 0$ is the elasticity of substitution. Cost minimization implies that the demand for intermediate good j is:

$$y_{j,t}(p_{j,t}) = (\frac{p_{j,t}}{P_t})^{-\eta} Y_t, \text{ where } P_t = (\int_0^1 p_{j,t}^{1-\eta} dj)^{\frac{1}{1-\eta}}.$$

Intermediate goods firms. Each intermediate firm j employs labor $n_{j,t}$ at the market wage w_t and operates the linear technology:

$$y_{j,t} = z n_{j,t},$$

where z is the productivity level.

I model price stickiness following Rotemberg (1982). Given past period price $p_{j,t-1}$, price adjustment cost Θ_t is a quadratic function of inflation and proportional to aggregate output Y_t:

$$\Theta_t = \frac{\theta}{2}(\frac{p_{j,t}}{p_{j,t-1}} - \bar{\pi})^2 Y_t,$$

where $\bar{\pi}$ is the steady state inflation rate.

I assume the government collects profits from intermediate goods firms and future profits are thus discounted at the risk-free interest rate. The Bellman equation for an intermediate goods firm is:

$$J_t(p_{t-1}) = \max_{p_t, y_t, n_t} y_t \frac{p_t}{P_t} - w_t n_t - \frac{\theta}{2}(\frac{p_t}{p_{t-1}} - \bar{\pi})^2 Y_t + \frac{1}{1+r_t} J_{t+1}(p_t),$$
$$s.t. \quad y_t = z n_t, \; y_t = (\frac{p_t}{P_t})^{-\eta} Y_t.$$

Imposing symmetric equilibrium s.t. $p_{t,j} = p_{t,j'}, \forall j, j'$, and denote the marginal cost of producing one unit of output as $mc_t = w_t/z$, I derive the following nonlinear New Keynesian Phillips curve:

Lemma 1. *The inflation rate $\pi_t = \frac{P_t}{P_{t-1}}$ is determined by the New Keynesian Phillips curve:*

$$\pi_t(\pi_t - \bar{\pi}) = \frac{\eta}{\theta}(mc_t - mc^*) + \frac{1}{1+r_t}(\pi_{t+1} - \bar{\pi})\pi_{t+1}\frac{Y_{t+1}}{Y_t}, \tag{3}$$

where $mc^* = \frac{\eta-1}{\eta}$ is the mark-up ratio at no-inflation stationary equilibrium.

$\bar{\pi}$ is the steady state inflation rate and is set to 1 throughout the paper. The recursive formula can be written in present-value form as:

$$\pi_t(\pi_t - \bar{\pi}) = \frac{\eta}{\theta}\sum_{s=0}^{\infty}(\prod_{\tau=0}^{s-1}\frac{1}{1+r_{t+\tau}})(mc_{t+s} - mc^*)\frac{Y_{t+s}}{Y_t},$$

which has the standard interpretation that inflation is positively related to the present value of marginal costs.

Finally, at symmetric equilibrium, the production function for final goods is reduced to:

$$Y_t = zN_t,$$

with profit $Profit_t = zN_t - w_t N_t - \Theta_t$, where N_t is the aggregate labor demand.

3.6 Government and Monetary Authority

The risk-free nominal interest rate i_t is determined by monetary policy. The monetary authority commits to the Taylor rule:

$$i_t = \bar{r} + \phi \log(\pi_t / \bar{\pi}) + \epsilon_t, \tag{4}$$

where $\bar{r}$ determines the steady state real interest rate. Shocks to monetary policy are modeled through ϵ_t and the main exercise of the paper studies the transition of economy after unexpected shocks of deterministic sequence ϵ_t.

The nominal interest rate transmits to real interest rate through the Fisher equation:

$$r_t = (1 + i_t)/\pi_{t+1} - 1. \tag{5}$$

The government collects profits from intermediate goods firms. The tax system consists of a proportional labor tax at rate τ_t and a lump sum transfer Tr_t to capture the progressive system. In addition, at every period t, the government issue bonds B_{t+1}. I maintain the convention that positive B_{t+1} denotes government saving and negative B_{t+1} denotes borrowing to be received/paid next period. The government collects profits from intermediate firms and labor taxes from households to finance government expenditure G_t and bonds interest. Government budget constraint is thus described by:

$$\frac{B_{t+1}}{1 + r_t} + G_t = B_t + Profit_t + \tau_t \int w_t n_t(e) e \, d\Phi_t - Tr_t.$$

Since the government is the only provider of liquidity in the economy, for the monetary authority to implement the Taylor rule, the government needs to change government bonds, or adjust government spending and transfer accordingly. An alternative interpretation is that the government and monetary authority commits to nominal interest rates through open market operations, and absorbs all the losses and gains from bonds interest.

I calibrate the level of government bonds to match US data at the stationary equilibrium and use government spending as a residual to clear government budget in the transition path.

3.7 Definition of Equilibrium

I define the following competitive equilibrium.

Definition 1. *Given B_0, initial probability measure Φ_0 over households' states (i, e, a, s), and sequence of exogenous policy shocks $\{\epsilon_t\}_{t=0}^{\infty}$, a competitive equilibrium is a sequence of (1) scalar prices $\{w_t, r_t, \pi_t\}_{t=0}^{\infty}$, (2) value and policy functions $\{W_t(i, e, a, s), m_t^{(d,a')}(i, e, a, s), n_t(e)\}_{t=0}^{\infty}$, (3) bond price functions $\{q_t^{(d,a')}(e, a, s)\}_{t=0}^{\infty}$, (4) score updating functions $\{\psi_t^{(d,a')}(e, a, s)\}_{t=0}^{\infty}$, (5) government policies $\{i_t, B_t, G_t, \tau_t, Tr_t\}_{t=0}^{\infty}$, (6) aggregate quantities $\{Y_t, C_t, N_t\}_{t=0}^{\infty}$, (7) probability measures over households' states $\{\Phi_t\}_{t=0}^{\infty}$, s.t.*

1. *$\{W_t(i, e, a, s), m^{(d,a')}(i, e, a, s)\}_{t=0}^{\infty}$ are solutions to households' decision problems. $n_t(e)$ is the decision rule for labor supply.*

2. *$q_t(a', e, a, s)$ satisfies financial intermediaries' zero-profit conditions.*

3. *$\psi_t^{(d,a')}(e, a, s)$ is consistent with Bayes rule and the transition of type switching shocks.*

4. *The Phillips curve is satisfied.*

5. *The Taylor rule and the Fisher equation are satisfied. Government budget constraint is satisfied.*

6. *Bonds market clears:*

$$\int \sum_{a'} m_t^{(0,a')}(i, e, a, s) q_t(a', e, a, s) a' d\Phi_t + \frac{B_{t+1}}{1 + r_t} = 0.$$

   Labor market clears:

$$Y_t = zN_t,$$
$$N_t = \int n_t(e) e d\Phi_t.$$

   Goods market clears:

$$C_t + G_t + \Theta_t = Y_t,$$

   where $C_t = \int \sum_{(d,a')} m_t^{(d,a')}(i, e, a, s) c_t^{(d,a')}(e, a, s) d\Phi_t$, and Θ_t is the price adjustment cost.

7. *Probability measures $\{\Phi_t\}_{t=0}^{\infty}$ are consistent with the transition induced by policy functions and exogenous shocks.*

I define a stationary competitive equilibrium that has certain equilibrium interest rate, labor income tax, and transfer as following:

Definition 2. *Fix $\epsilon_t = 0\ \forall t$, a stationary equilibrium $SCE(r, \tau, Tr)$ is a competitive equilibrium with time-invariant equilibrium objects.*

The government can implement any $SCE(r, \tau, Tr)$ by adjusting government spending accordingly. I establish the following existence theorem when the scale parameter σ_ε of the type I extreme value shock is large enough[7].

Proposition 1. *For any $r > 0, \tau \in [0, 1)$ and $Tr \geq 0$, $\exists \sigma_\varepsilon^*(r, \tau, Tr)$ s.t. $\forall \sigma_\varepsilon > \sigma_\varepsilon^*(r, \tau, Tr)$, a stationary equilibrium $SCE(r, \tau, Tr)$ exists.*

The role of the type I extreme value shock is to "smooth out" the policy function. The challenge in the existence proof is to show the policy function and implied bond and posterior functions are uniformly Lipschitz continuous in credit score and continuous in other equilibrium objects even the feasible set of actions can vary "discontinuously".

4 Analytical Characteristics of the Decision Problem

Understanding how credit is rationed in the model is crucial in understanding how monetary policy passes through extending credit. The credit limits in the model arise both due to limited enforcement a la Eaton and Gersovitz (1981) and adverse selection a la Stiglitz and Weiss (1981). With further assumptions on the model parameters, several analytical results can be derived to illustrate the mechanism.

Assumption 1. *The type switching shock is transitory., i.e., $\Omega(g|g) = \Omega(g|b) = 1/2$.*

With this assumption, risk type is not persistent and credit score is not relevant for bond price in equilibrium. However, properties of households' behaviors can be characterized to help understand how bond price functions are shaped in more general cases.

First, similar to Arellano (2008) and Chatterjee et al. (2007), the default set (of the transitory preference shocks) is expanding with the loan size, and the default probability is increasing in loan size. In equilibrium, by zero profit conditions of financial intermediaries, the loan price is inversely linked to the expected default probability, so bond price is decreasing in loan size.

Proposition 2. *With Assumption 1, in any stationary equilibrium, the default probability is increasing in loan size, i.e. $m^{(1,0)}(i, e, \tilde{\tilde{a}}, s) \geq m^{(1,0)}(i, e, \tilde{a}, s), \forall \tilde{\tilde{a}} \leq \tilde{a} < 0, \forall i, e, s$. The bond price is decreasing in loan size, i.e. $q(\tilde{\tilde{a}}', e, a, s) \leq q(\tilde{a}', e, a, s), \forall \tilde{\tilde{a}}' \leq \tilde{a}' < 0, \forall e, a, s$.*

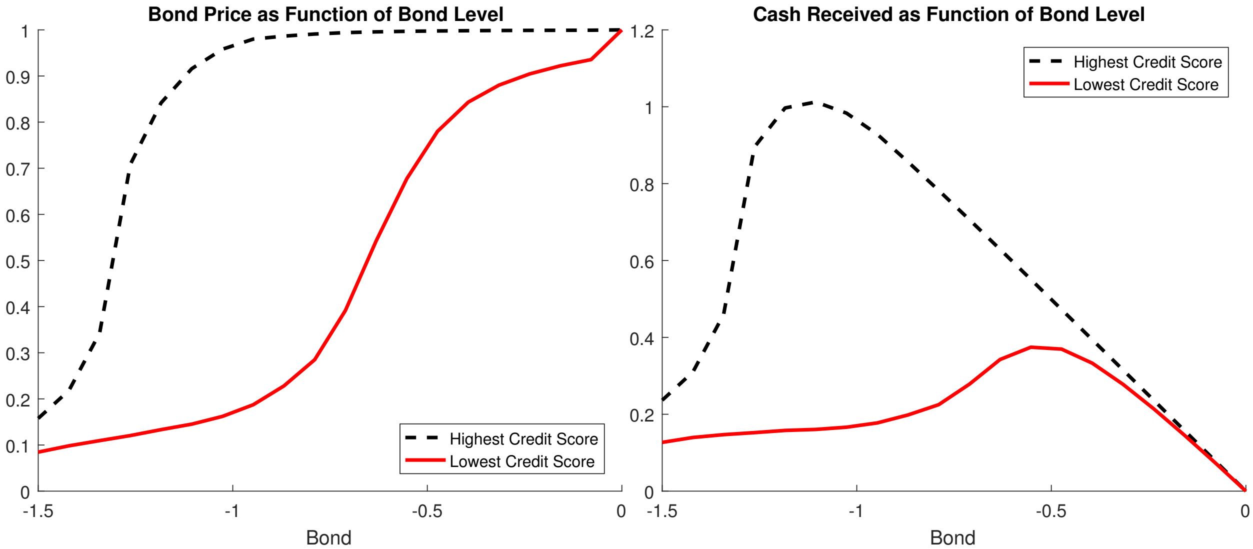

Figure 2: Mechanisms for Credit Rationing: Limited Enforcement

Notes: The left panel depicts the bond price function $q(a', \cdot)$ for different bond levels a' at median labor efficiency shock and zero current bond holding, for $\underline{s}$ and $\bar{s}$. The right panel depicts the cash received from taking negative bond, i.e., $-a'q(a', \cdot)$ varying bond levels at median labor efficiency shock and zero current bond holding, for $\underline{s}$ and $\bar{s}$.

The implications of Proposition 2 can be best illustrated in Figure 2[8]. The left panel depicts the discount price for loan as a function of loan size. As loan size increases (bond decreases), to account for the additional default risk, financial intermediaries offer a lower discount price. The cash from a bond contract received in the current period is equal to the product of loan value and discount price, and reaches maximum as the discount price decreases, depicted in the right panel.

The most important property introduced by asymmetric information which serves as the core of the transmission mechanism is that taking a larger loan signals a risky type, characterized by the following proposition.

Proposition 3. *With Assumption 1, in any stationary equilibrium, taking a larger loan signals the borrower is more likely to be a risky type, i.e.,* $\xi^{(0,\tilde{\tilde{a}}')}(e,a,s) \leq \xi^{(0,\tilde{a}')}(e,a,s), \forall \tilde{\tilde{a}}' \leq \tilde{a}' < 0, \forall e, a, s.$

Intuitions of Proposition 3 can be illustrated by Figure 3[9]. As shown in the left panel, due to the transitory preference shocks, households' policy functions for inter-temporal

[7] All proofs are in appendix B.

[8] Figure 2 is actually constructed with the benchmark calibration (instead of imposing Assumption 1). The qualitative properties derived under Assumption 1 still hold.

[9] Again, Figure 3 is constructed with the benchmark calibration.

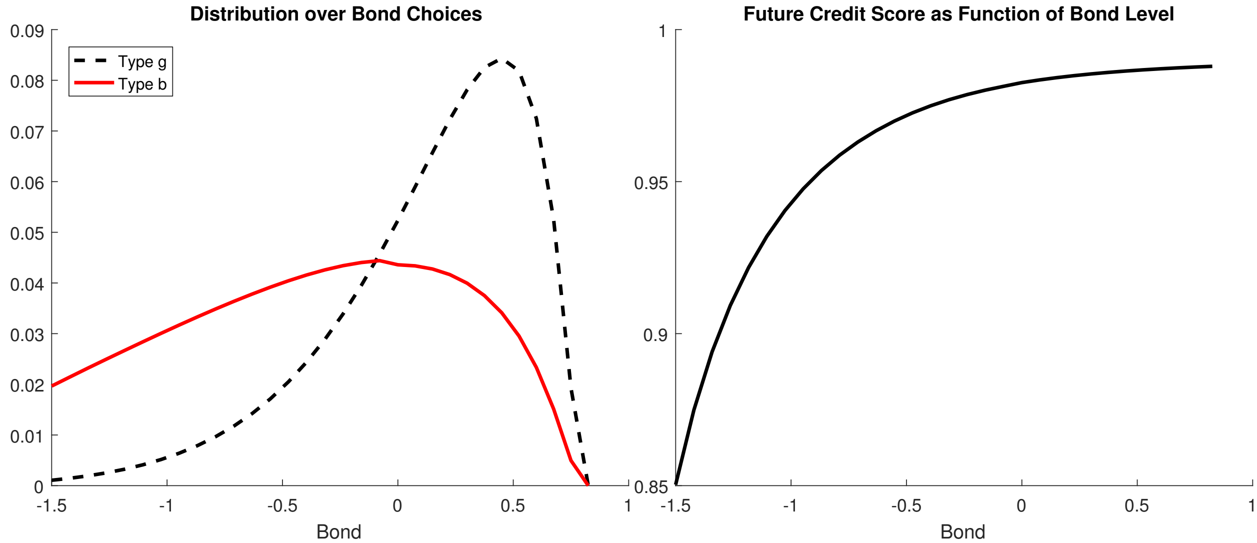

Figure 3: Mechanisms for Credit Rationing: Adverse Selection

Notes: The left panel depicts the bond policy function function $m^{(0,a')}(\cdot)$ for different bond levels a' at median labor efficiency shock, zero current bond holding and $\bar{s}$, for the safe and risky types. The right panel depicts the credit score (belief) updating function $\psi^{(0,a')}(\cdot)$ for different bond levels a' at median labor efficiency shock, zero current bond holding and $\bar{s}$.

decisions can be described by distribution over bond choices. Since the safe type is more patient ($\beta_g > \beta_b$), the mass put on higher bonds is greater, and the mass on lower bonds is lesser. As a consequence, taking a larger loan (a lower bond) signals the borrower is more likely to be the risky type, as shown in the right panel.

Finally, the safe type is less likely to declare bankruptcy if the utility cost of default is large enough.

Proposition 4. *With Assumption 1, for any* $r > 0, \tau \in [0,1), Tr \geq 0$, *and* $\phi_b \in (0,1)$, $\exists \phi^*(\phi_b; r, \tau, Tr) > 0$, *s.t.* $\forall \phi_g \in (0, \phi^*(\phi_b; r, \tau, Tr))$, *in any stationary equilibrium* $SCE(r, \tau, Tr)$ *the default probability satisfies* $m^{(0,1)}(b, e, a, s) \geq m^{(0,1)}(g, e, a, s), \forall a < 0, e, s$.

Putting all pieces together, credit limits first arise because households with larger loans are more likely to default (Proposition 2). Second, since the risky type is more likely to default (with the condition in Proposition 4 satisfied), credit limit is tighter if the posterior probability that a borrower is a risky type is higher, but taking a larger loan indeed signals a risky type (Proposition 3), and consequently credit is further rationed.

A shock lowering the risk-free interest rate pools the safe and risky type together as borrowers, improves credit scores for borrowers, and alleviates credit rationing due to adverse selection. This is characterized by Proposition 5.

Proposition 5. *With Assumption 1, suppose households face different bond price function $\tilde{q}$ other than the equilibrium one q^* for one period, s.t. $\tilde{q} = (1+\Delta)q^*$ with $\Delta > 0$, then the posterior probability of being a good type is higher under the one-period bond price $\tilde{q}$, i.e., denote the posterior probability function under one-period bond price q as $\xi^{(d,a')}(e,a,s;q)$, then $\xi^{(0,a')}(e,a,s;\tilde{q}) \geq \xi^{(0,a')}(e,a,s;q^*)$.*

Notice the change in risk-free interest rate exactly changes the bond price function proportionally if the default probability were held constant. Therefore, the lower real interest rate following accommodative monetary policy shock shifts the equilibrium bond price function up proportionally. Following Proposition 5, this improves credit score of borrowers, and is precisely why credit price improves more for ex-ante more risky households after the monetary policy shock. In the actual transitional equilibrium to analyze, the bond price function shifts up for multiple periods and the default probability changes, but the intuition established here holds. We will return to the underlying mechanism described by Proposition 5 in the study of the transition path after monetary policy shocks.

5 Mapping the Model to US Data

5.1 Model Specification

The utility function over consumption and leisure bundle is specified as:

$$u(\tilde{c}) = \frac{\tilde{c}^{1-\sigma}}{1-\sigma}.$$

The consumption leisure trade-off is specified with constant Frisch elasticity over labor supply:

$$c - v(n) = c - \chi \frac{n^{1+1/\nu}}{1+1/\nu},$$

where the parameter ν governs the wage elasticity of labor supply and χ governs the level of labor supply.

The transition matrix Γ for labor efficiency shocks is discretized from an AR1 process, with persistence parameter ρ and unconditional variance σ_e^2:

$$\log(e') = (1-\rho)\log(e) + u',$$

where $u' \sim N(0, \sigma_u^2)$. The process is discretized with 7 states over ± 3 unconditional standard deviations.

The transition matrix Ω for the type switching shock is specified as

$$\Omega = \begin{bmatrix} \Omega_{bb} & 1 - \Omega_{bb} \\ 1 - \Omega_{gg} & \Omega_{gg} \end{bmatrix},$$

I estimate the model to match several aggregate moments and cross-sectional facts in the consumer credit market. First, I define the model counterparts of several data statistics.

5.2 Derive Model Statistics

Credit limits. The model endogenously generates a credit limit for each household that is a function of observable characteristics (e, a, s). It is the maximum level of borrowing that a household can receive across all loan contracts, described by $CL(e, a, s; q_t) = \max\limits_{a' \in \mathbb{A}, a' < 0} -q_t(a', e, a, s)a'$, where q_t is the equilibrium bond price function.

Marginal propensity to borrow out of extended credit. Agarwal et al. (2015b) estimate the marginal propensity to borrow out of extended credit with a regression discontinuity approach, exploiting the fact that credit limits are not assigned continuously in credit scores. Two consumers with close credit scores may face very different credit limits because their credit scores lie on different sides of certain threshold that lenders use to assign credit limits. Comparing the differences in their credit limits and credit card utilization after origination, Agarwal et al. (2015b) estimate how much additional borrowing is due to the exogenous increase in credit limits, for card holders with different credit scores.

To generate a similar concept of marginal propensity to borrow after an "exogenous" increase in credit limits from the model, consider the following thought experiments. Denote the bond price function at stationary equilibrium as $q^*(a', e, a, s)$, and consider a one-time deviation of the bond price function from the stationary equilibrium to

$$\tilde{q}(a', e, a, s; \Delta) = \begin{cases} (1 + \Delta) q^*(a', e, a, s), & \forall a' < 0, \\ q^*(a', e, a, s), & \forall a' \geq 0. \end{cases}$$

We first have credit limits $CL(e, a, s; q)$ defined for both equilibrium bond price function q^* and the deviated function $\tilde{q}$.

Then I ask if a household faces this deviated bond price function for one period, with the continuation value held unchanged as in the stationary equilibrium, what is her optimal decision. This experiment can be interpreted as that with zero probability financial intermediaries make the mistake by setting the bond price function to be $\tilde{q}$ instead of q^*, so even though the household faces the bond price function different than the equilibrium one at current period, she understands with zero probability it will happen again in the future.

To be specific, the household solves the following problem:

$$\max_{(d,a')\in M(e,a,s;q)} U^{(d,a')}(i,e,a,s;q) + \varepsilon^{(d,a')} + \beta_i \sum_{i'\in\{b,g\}} \Omega(i'|i) \sum_{e'} \Gamma(e'|e)\, W(i',e',a',\psi^{(d,a')}(e,a,s)),$$

where W is the equilibrium value function, and $U^{(d,a')}(i,e,a,s;q)$ is the period return function under bond price function q, for $q = q^*$ and $q = \tilde{q}$. Denote the policy function to the optimization problem by $m^{(d,a')}(i,e,a,s;q)$, then the expected amount of borrowing from households with state variable (i,e,a,s) can be expressed as:

$$A(i,e,a,s;q) = -\sum_{(0,a')} m^{(0,a')}(i,e,a,s;q)q(a',e,a,s)a',$$

The marginal propensity to borrow out of extended credit for households with state variable (i,e,a,s) is then defined as the ratio of change in borrowing and change in credit limit:

$$MPB(i,e,a,s;\Delta) = \frac{A(i,e,a,s;\tilde{q}) - A(i,e,a,s;q^*)}{CL(e,a,s;\tilde{q}) - CL(e,a,s;q^*)}.$$

I set Δ to be $1e-6$ to evaluate the statistics after solving the stationary equilibrium.

Statistics by credit score groups. Credit scores in the data are the FICO scores, measured by integers ranging from 350 to 850. The model counterpart of credit scores is the prior probability s which lies in a proper subset of $[0,1]$. Since credit scores in the model and data are under different ordinal metrics, I transform both model and data statistics by percentile of credit scores. This shows one of the advantages of the current general equilibrium framework. With the invariant probability measure over endogenous state variables solved as part of the stationary equilibrium, all model statistics can be derived for different credit score percentiles.

5.3 Parameters

I set the model period to be one quarter. Parts of the model parameters are set exogenously, reported in the first part of Table 1. In particular, the inter-temporal elasticity in the CRRA utility function σ is set to 1.5, a standard value used in the literature (e.g. Smets and Wouters (2007) report point estimate of 1.47 for the "Great Moderation" period). The Frisch elasticity of labor supply ν is set to 0.5, a value that lies in the middle of the micro and macro estimates surveyed in Keane and Rogerson (2011). The persistence of labor efficiency shocks is set to 0.935 and the unconditional variance of labor efficiency shocks is set to 0.01, which are values estimated in Guvenen and Smith (2014), converted to quarterly frequency. The elasticity of substitution in the final goods production is set to 10, the coefficient determining cost of price adjustment is set to 100, and the responsiveness to inflation in the Taylor rule is set to 1.25. These are common values in the literature and used in Kaplan et al. (2015). The intercept of Taylor rule determining the quarterly risk free interest rate is set to 0.75%. The flat rate of labor income tax is set to 25%.

The remaining parameters are estimated to match several aggregate moments and cross-sectional facts in the consumer credit markets using Simulated Methods of Moments (McFadden (1989)). Table 1 reports the parameters and the targeted moments. All parameters are estimated jointly. The most closely associated moments for each parameter are reported. I calibrate the government transfer such that half of households receive positive net transfer as in Kaplan et al. (2015). The two additional aggregate moments targeted include the mean work hours of 0.33 and the government debt to output ratio of 0.47 (the average value from 1970 to 2007 for the US economy).

The unconventional parameters specific to the model, including the discount factors, default costs, and switching probability of the two types, are disciplined by the 12 moments of the cross-sectional facts in the consumer credit market, as shown in Figure 4. These moments are taken from Agarwal et al. (2015b), which are derived using the Credit Card Metrics data covering the universe of US credit card accounts between January 2008 and December 2014. These moments include the fraction of accounts delinquent with more than 90 days past due, credit limits on credit cards, and marginal propensities to borrow out of extended credit. For the fraction of accounts delinquent, I associate it with the default rate in the model. For the credit limits, I match the ratio between each credit score group and the lowest credit score group, since in the data I only observe the credit limits on single credit card but not the total. For the marginal propensity to borrow out of extended credit, I construct model statistics following the procedure described in section 5.2.

The data moments are reported for the 4 credit score groups which correspond to 28%, 16%, 19% and 37% of the population, as shown in the first panel of Figure 4. Inverting the cumulated distribution function of credit score at the stationary equilibrium, these 4 credit score groups correspond to the range $[0.1, 0.94)$, $[0.94, 0.97)$, $[0.97, 0.98)$, and $[0.98, 0.99)$ of model credit score, respectively. The model statistics for each credit score group are thus constructed by averaging across households within each range.

All parameters are jointly calibrated. However, it is important to understand what aspect of the data speaks most to a certain parameter. First of all, the disutility in working captured by parameter χ is associated with the average hours worked. The productivity level z is chosen to normalize the median wage to 1, mostly due to computation convenience.

The first empirical regularity, the extent to which the default rate varies with credit score, puts discipline on the parameters governing default disutility of the two types. Intuitively, a higher discount of utility upon default prevents households declaring bankruptcy often. The calibrated utility discount for the safe type is 0.018, and for the risky type is 0.9. At first glance, the difference may look huge: the safe type has to lose 98% of current consumption to bring the default rate of the highest credit score group close to data. But if one understands the model is calibrated at the quarterly level, and all the explicit utility cost incurs at current period, then the 98% of consumption should be viewed as the sum of discounted costs that would incur at many periods into the future. For example, in the US, the bankruptcy flag will stay on a consumer's credit report for 10 years if he declares bankruptcy. These are features that are not directly captured by the static utility cost (though the endogenous reputation erosion in the model does account for part of the future cost), which is responsible for the calibrated low value of the utility discount parameter for the safe type.

The second empirical regularity, the extent to which the marginal propensity to borrow and spend varies with credit score, puts discipline on the two discount factors. Though discount factors are frequently used in the literature to match marginal propensities to spend *out of transitory income* (e.g. Auclert (2014)), it is not obvious why they are essential in determining the marginal propensities to spend *out of extended credit*. In the model, discount factors are associated with marginal propensities to spend out of extended credit for two reasons. First, given the current bond holding, a lower discount factor means valuing future consumption less and higher probability put on the lower bond choices. An increase in discount prices for negative bonds (a credit extension) thus results in larger changes in the expected level of bond holdings, which manifests itself as larger amounts to be borrowed and spent out. Second, at stationary equilibrium, households with lower discount factors are those with larger loans, which have even higher tendency to put larger probability on lower bond choices. The calibrated discount factor for the safe type is 0.99, and for the risky type is 0.54. The large differences mainly come from the starking differences in marginal propensities across credit score groups: while the lowest credit score group spends 60% of the extended credit within a quarter, the highest credit score group virtually does not respond to credit extension at all.

The third empirical regularity, the extent to which credit limit varies with credit score, puts discipline on the dispersion parameter of the transitory preference shock, which eventually determines the degree of adverse selection in the model. The dispersion parameter essentially determines how alike the risky and the safe types behave. If the dispersion is low and bond choices are distinct, households' types will be revealed by the bond choices immediately. In this scenario, the prior information, credit score, will not be used by financial intermediaries for credit pricing. On the contrary, if the dispersion is high and households' bond choices look alike, the credit score becomes more valuable in predicting the type. The dispersion parameter is crucial in determining the effectiveness of transmission mechanism of monetary policy through changing lenders' beliefs. If adverse selection is light and lenders can infer households' types perfectly, there is no role that monetary policy would affect the credit price through changing lenders' beliefs.

Finally, the two type switching probabilities determines the fraction of patient households, which affects the aggregate bond holdings of households, and on the other side, the level of government debt in the economy.

The current paper, to my best knowledge, is the first one to bring a model calibrated to the *cross-section* facts of unsecured credit. One reason why this has not been done is because earlier works like Chatterjee et al. (2007) and Athreya et al. (2012) do not have a notion of "credit score" that associated data moments with household default risk. In these traditional works, models are taken to match the aggregate default rate, credit price, and bond level. Chatterjee et al. (2011), with which the current paper shares same notion of "credit score", explores qualitative properties of the cross-sectional distribution produced by models, but does not attempt to quantitatively match the data moments. In the current paper, I show that for the model to produce the correlation between default rate and credit score close to data, it is indispensable to assume different costs of default and allow the cost of default to be correlated with the discount factor. As growing empirical works like Agarwal et al. (2015b) come out, I expect more research on understanding the implications of these cross-sectional facts on consumer credit. The current paper is a first step toward this goal.

Parameter	Description	Value	Target	Model
Exogenous:				
σ	CRRA coefficient	1.5		
ν	Frisch elasticity of labor supply	0.5		
ρ	Persistence of labor efficiency shocks	0.935		
σ_e^2	Unconditional variance of labor efficiency shocks	0.01		
η	Elasticity of stitution of intermediate goods	10		
θ	Price adjustment cost	100		
ϕ	Taylor rule in response to inflation	1.25		
$\bar{r}$	Steady state risk free interest rate	0.75%		
τ	Labor income tax rate	25%		
Calibrated:				
χ	Weight of labor in utility function	21.86	Average hours = 0.33	0.33
z	Productivity level	3.39	Median wage = 1	1
β_g	Discount factor of type g	0.99	$B/Y = 0.47$	0.48
β_b	Discount factor of type b	0.54		
ϕ_g	Default utility discount of type g	0.018		
ϕ_b	Default utility discount of type b	0.9	Cross-sectional distribution	
Ω_{gg}	Type switching probability, g to g	0.99		
Ω_{bb}	Type switching probability, b to b	0.90		
σ_ε	Scale paramter of T1EV preference shock	2.21		

Table 1: Model Parameter Value

6 Transition Path after Unexpected Shocks to the Taylor Rule

I model monetary policy shocks as unexpected shocks to the Taylor rule specified in Equation (4). At period 0, the economy is at the stationary equilibrium. At period 1, a sequence of deterministic shock $\{\epsilon_t\}_{t=1}^{T}$ is announced by the monetary authority. The shock is generated by the deterministic first-order autoregressive process $\epsilon_{t+1} = \rho_r \epsilon_t$ with persistence $\rho_r = 0.8$ and initial size $\epsilon_1 = -0.25\%$, which lowers the risk-free nominal interest rate by 25 basis point on impact. I study the transition path of the economy after the shock hits. I assume in order to implement the Taylor rule, the government keeps balanced budget by adjusting government spending accordingly, keeping government bonds and transfer at the stationary equilibrium level.

6.1 Aggregate and Heterogeneous Consumption Responses

The left panel of Figure 5 presents the exogenous sequence of shocks to the Taylor rule, and endogenous responses of risk-free real interest rate and inflation rate. The

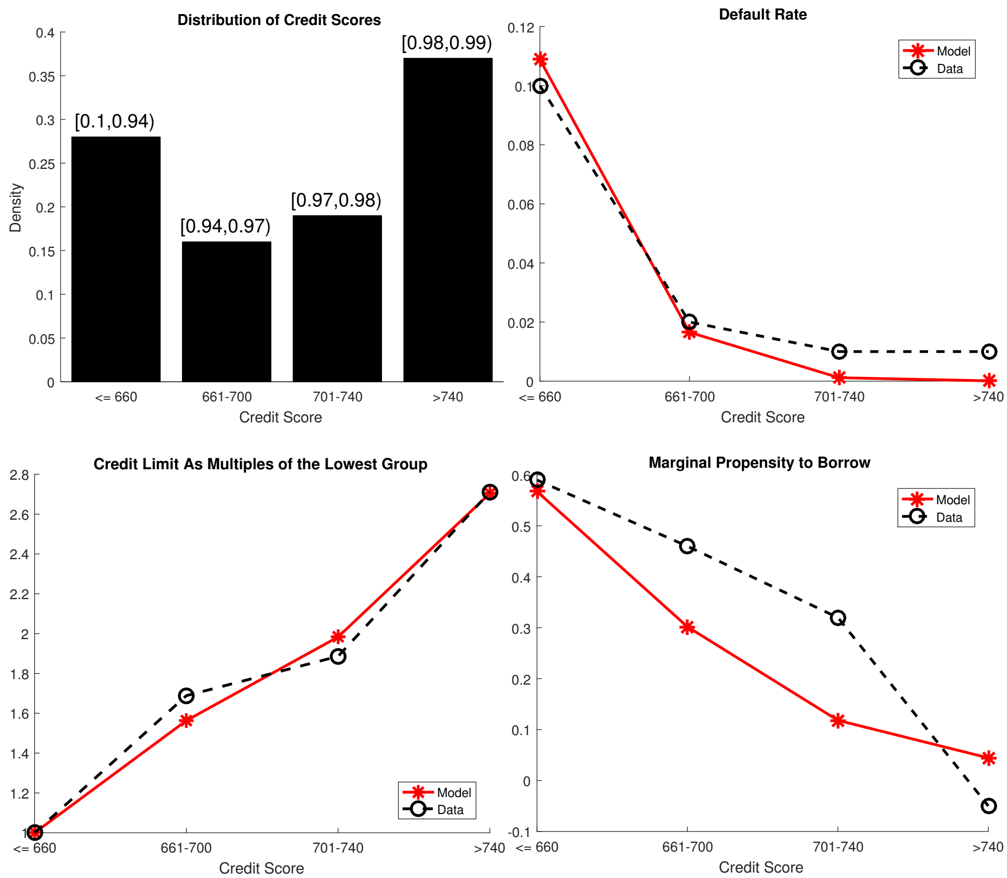

Figure 4: Statistics by Credit Score Groups, Model v.s. Data

Notes: The four credit score groups each corresponds to 28%, 16%, 19% and 37% of the population. Data statistics are from Agarwal et al. (2015b). Model statistics are computed by averaging statistics across simulated samples within each percentile range.

initial response in consumption and how it is able to drive output response is similar to the standard New-Keynesian mechanism: the lowering of risk-free nominal interest rate transmits to higher prices for discount bonds at each loan level (and lower rate of return for deposits) since financial intermediaries are pricing bonds competitively. Therefore, consumers demand more final goods by saving less or borrowing more due to the intertemporal substitution effects. If there were no adjustment costs for prices of goods, intermediate goods firms would raise prices accordingly, which would increase inflation and decrease the real interest rate, leaving the real part of the economy unchanged. However, due to price stickiness, prices are raised not enough to cancel out the initial fall in nominal interest rate, leaving real interest rate lower than the stationary equilibrium level. Consistent with the path of real interest rate, consumption is higher than the stationary equilibrium level and wage rises accordingly to induce more labor supply to

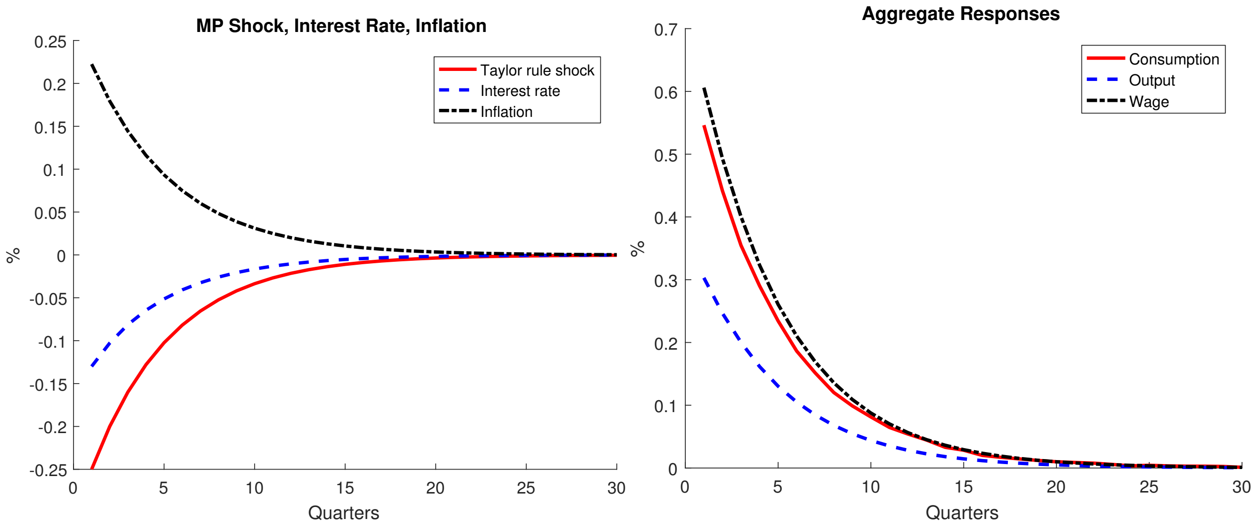

Figure 5: Monetary Policy Shocks and Aggregate Responses

clear the goods market. The general-equilibrium effects of rising wage further amplifies the consumption responses[10], a point made by Kaplan et al. (2015). The responses of aggregate output, consumption, and wage are presented in the right panel of Figure 5.

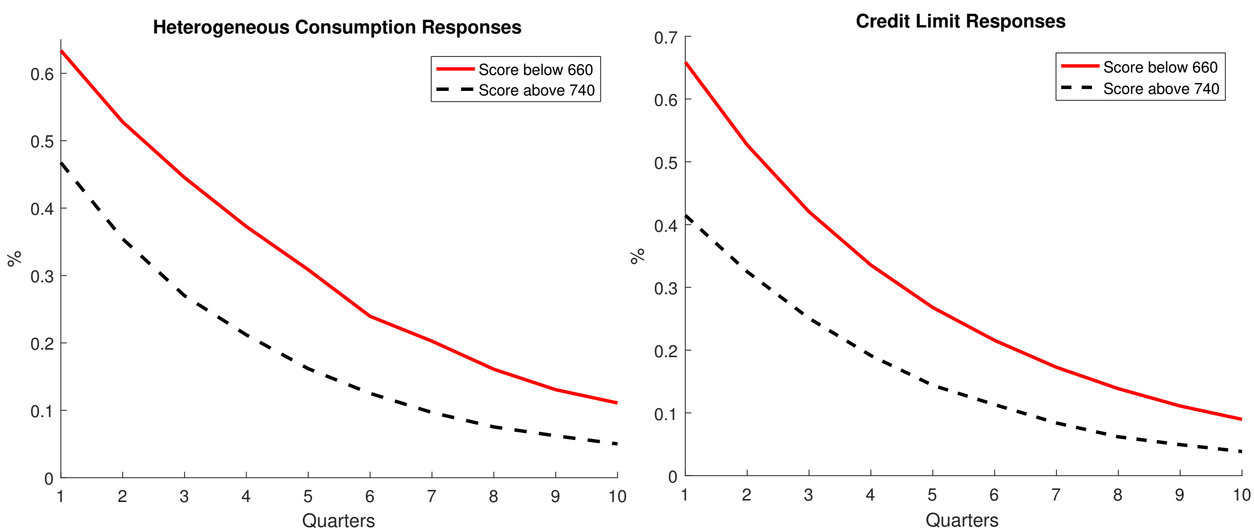

Figure 6: Heterogeneous Consumption and Credit Limit Responses

The left panel of Figure 6 presents the heterogeneous consumption responses, measured as percentage deviation from the stationary equilibrium level, tracking households who are in the lowest and highest credit score groups at the stationary equilibrium[11].

[10]Since I balance government budget using government spending as the residual, in the model there are no general equilibrium effects from changes in government transfer or government bond.

[11]Recall model credit score is converted to data credit score following procedures described in Section

The consumption response for the lowest credit score group is 0.63% on impact, higher than the response for the highest credit score group of 0.47%. Table 2 presents the consumption responses tracking different credit score groups measured by absolute level and percentage deviation. As shown on the second row of the table, the consumption responses measured by percentage deviation are monotonically decreasing in credit scores. I compute the cumulated responses along the transition path discounted by real interest rates, and the results are presented in the fourth row of Table 2. As shown, the cumulated responses are also monotonically decreasing in credit scores and the differences are starker: the cumulated response for the lowest credit score group is 3.45% of the stationary equilibrium level while the cumulated responses for the highest credit score group is only 1.99%.

	≤ 660	$661-700$	$701-740$	> 740	Average	Δ Low - High
			Endogenous Credit Scoring Function			
On Impact	0.00446	0.00534	0.00477	0.00397	0.00448	0.00050
% On Impact	0.63%	0.59%	0.54%	0.47%	0.54%	0.16%
Cumulated Response	0.0274	0.0294	0.0238	0.0159	0.0228	0.0115
% Cumulated Response	3.45%	3.20%	2.74%	1.99%	2.75%	1.46%
			Fixed Credit Scoring Function			
On Impact	0.00358	0.00419	0.00399	0.00329	0.00365	0.00029
% On Impact	0.50%	0.47%	0.45%	0.39%	0.44%	0.11%
Cumulated Response	0.0172	0.0208	0.0194	0.0158	0.0177	0.0014
% Cumulated Response	2.40%	2.33%	2.21%	1.87%	2.15%	0.54%

Table 2: Consumption Responses of Different Credit Score Groups

Notes: The four credit score groups each corresponds to 28%, 16%, 19% and 37% of the population."On impact" corresponds to the deviation from stationary equilibrium value at the period when the shock hits. "% on impact" corresponds to the percentage deviation. "Cumulated response" corresponds to sum of deviations discounted by real interest rates through the transition path. "% cumulated response" corresponds to the cumulated response as percentage of the stationary equilibrium level.

The consumption response is higher for households with lower credit scores, because first, they have higher marginal propensities to spend out of extended credit, a property arising from heterogeneous discount factors disciplined by the cross-sectional distribution in the calibration. Second, the response of credit limit is greater for households with lower credit score, as shown in the right panel of Figure 6.

Why is credit extension stronger for households which have higher default risk ex-

5.3, and the two groups correspond to the lowest 28% and the highest 37% of credit score, respectively.

ante? Recall the equations that characterize the bond price function and the credit limit:

$$q_t(a', e, a, s) = \frac{\sum_{e' \in \mathbb{E}} \Gamma(e'|e)\ pr_{t+1}(e', a', \psi_t^{(0,a')}(e, a, s))}{1 + r_t}, \tag{6}$$
$$CL(e, a, s; q_t) = \max_{a' \in \mathbb{A}, a' < 0} -q_t(a', e, a, s)a',$$

where pr_{t+1} is the probability of repayment at period $t+1$, ψ_t is the credit scoring function, and r_t is the risk-free interest rate. When r_t decreases, holding the equilibrium default probability function pr_{t+1} and credit scoring function ψ_t unchanged, q_t would change proportionally to the change in $1/(1+r_t)$, and the credit limits would change in equal proportion $1/(1+r_t)$. One thus would not expect to see heterogeneous changes in credit limits measured in *percentage* deviation. Indeed, the key mechanism driving heterogeneous credit limit responses lies in the response of credit score (belief) updating function ψ_t.

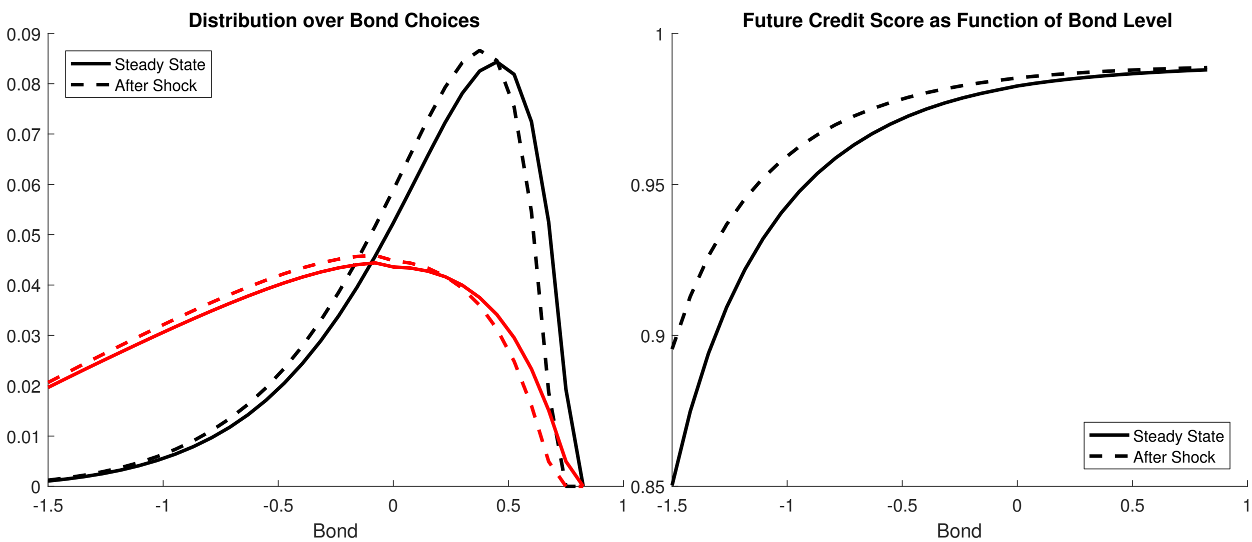

Figure 7: Mechanism of the Credit Channel: Response of the Credit Scoring Function

Notes: The left panel depicts the distribution of asset choices of the two types of households with median labor efficiency level, zero bond, and highest credit score. The solid curves correspond to type g and dashed curves correspond to type b. The right panel correspondingly depicts the credit scoring function at different loan sizes for the median labor efficiency level, zero bond, and highest credit score.

Figure 7 illustrates the response of the credit scoring function ψ_t at the period when the shock hits compared with that at the stationary equilibrium. To understand the response, the left panel depicts the distribution of bond choices for both types of households with median labor efficiency, zero asset, and highest credit score. The solid curve corresponds to the safe type and the dashed curve corresponds to the risky type. As characterized in Proposition 3[12], in equilibrium, the risky type is more likely to choose lower bonds compared to the safe type. Therefore, choosing lower bonds signals that the household is likely to be a risky type and lowers future credit score. When the monetary policy shock hits which lowers the real interest rate in equilibrium, for both types the probability of choices on negative bonds increases due to inter-temporal substitution effects, as illustrated by the shifts from black curves to red curves in the left panel. Since the safe type is more patient, initially the her inter-temporal policy function allocates a lesser probability on the negative bond choice, hence the relative increase in probability on negative bond choice is greater for her than that of the risky type. In the Bayesian updating specified in Equation (2) rewritten here

$$Pr_t(g|x,(d,a')) = \frac{s}{s + \frac{m_t^{(d,a')}(b,x)}{m_t^{(d,a')}(g,x)} \cdot (1-s)},$$

the posterior is determined by the likelihood *ratio* of choices between the two types. As the relative increase in the probability on bond choice of the safe type $m_t^{(d,a')}(g,x)$ is greater than that of the risky type $m_t^{(d,a')}(b,x)$, the ratio $\frac{m_t^{(d,a')}(b,x)}{m_t^{(d,a')}(g,x)}$ decreases and the posterior probability of being a safe type increases[13]. This results in the upward shifting of credit scoring function as shown by the change from the black curve to the red curve in the right panel of Figure 7.

The above intuition can be best formed through the following thought experiment. Suppose the risky type has zero discount factor and receives no transitory preference shocks, then he will always borrow up to the credit limit if he borrows at all. In normal times when the interest rate is high, the safe type is less likely to borrow and financial intermediaries expect a household which borrows to the credit limit is the risky type. The decline in interest rate following the monetary policy shock encourages the safe type to borrow and does not change the risky type's borrowing behavior (he still borrows to the credit limit), the pool of households which demand credit up to the limit becomes on average "safer" and credit price improves.

Since a lower credit score corresponds to higher prior probability of being a risky type, households with lower credit scores initially suffered from even more severe ad-

[12]Proposition 3 is shown under more restrictive assumptions. However, the properties hold for more general cases as in the current calibrated model.

[13]Indeed, the ratio decreases regardless of whether $m_t^{(d,a')}(g,x)$ and $m_t^{(d,a')}(b,x)$ increase or decrease. The logic is that initially the safe type puts greater probability on positive bond choice, hence whenever $m_t^{(d,a')}(g,x)$ and $m_t^{(d,a')}(b,x)$ decrease, the relative decrease in $m_t^{(d,a')}(g,x)$ would be smaller than $m_t^{(d,a')}(b,x)$, leading again a decrease in the ratio. See Proposition 5 and the proof.

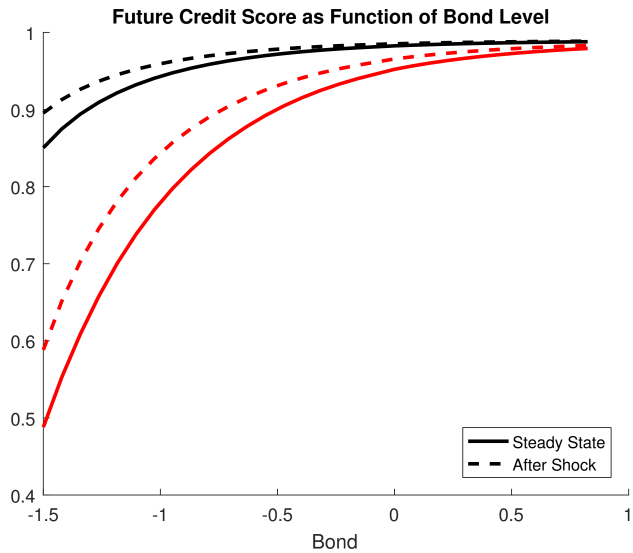

Figure 8: Response of the Credit Scoring Function, Highest v.s. Lowest Current Scores

Notes: The two higher curves depict the credit scoring function for the median labor efficiency level, zero bond, and highest credit score, before and after the monetary policy shock. The two lower curves depict the credit scoring function for the median labor efficiency level, zero bond, and lowest credit score.

verse selection. This can be illustrated in Figure 8. The two pairs of curves describe the change in credit scoring function for households initially in the highest and lowest credit score groups. The two higher curves correspond to the case when the current credit score is high, taken from the right panel of Figure 7. The two lower curves correspond to the case when the current credit score is low. As shown, the change in future credit score after the monetary policy shock is greater and consequently credit price improves even more when the current credit score is lower.

6.2 Model-based Decompositions

How important is the endogenous response of credit scoring function in driving heterogeneous consumption responses? I consider the following counter-factual experiment. Suppose financial intermediaries ignore the changes in borrowing behaviors when making inferences on households' types, then the mechanism through the endogenous response of the credit scoring function is "turned off". To implement this experiment, I construct a sequence of counter-factual bond price functions as

$$\tilde{q}_t(a', e, a, s) = \frac{\sum_{e' \in \mathbb{E}} \Gamma(e'|e) \; pr_{t+1}(e', a', \psi_*^{(0,a')}(e, a, s))}{1 + r_t},$$

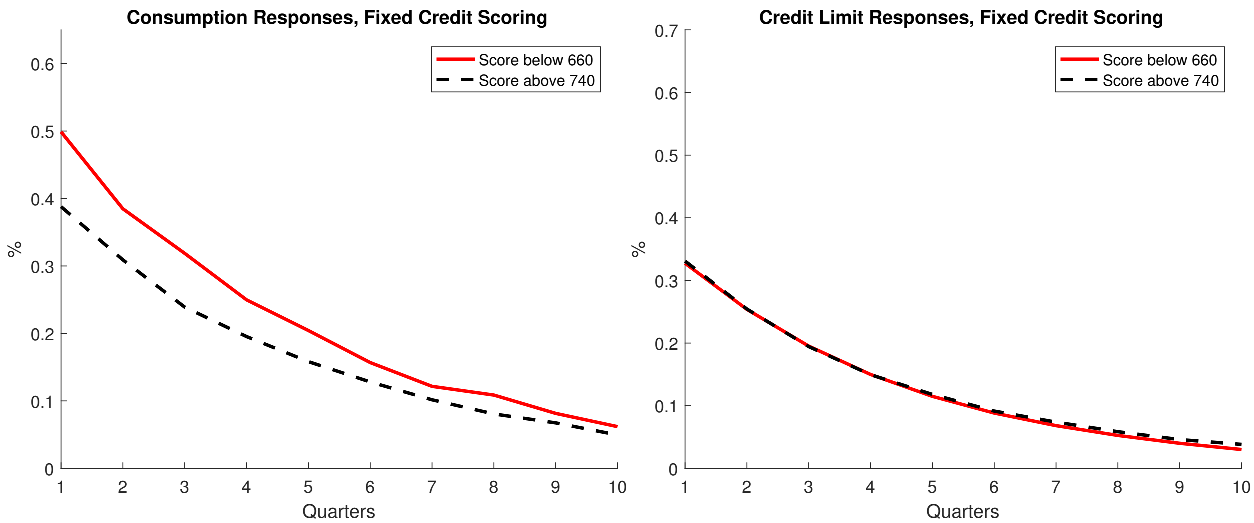

Figure 9: Responses with the Change in Credit Score (Belief) Updating Turned Off

where ψ_* is the credit scoring function fixed at the stationary equilibrium. That is, financial intermediaries correctly anticipate the change in real interest rate r_{t+1} and repay probability pr_{t+1}, but ignore the change in the credit scoring function. I then solve households' problems given these counter-factual bond price functions and the equilibrium path of wages. The generated counter-factual consumption responses are depicted in the left panel of Figure 9. Compared with the left panel of Figure 6, consumption responses for both groups are of smaller magnitudes, and the difference between the two groups shrinks. As reported in Table 2, the difference in consumption responses between the lowest and highest credit score groups is 31% (5% of 16%) lower measured on impact, or 63% (0.92% of 1.46%) lower measured by cumulated response.

The difference in consumption responses shrinks precisely because the difference in credit limit response drops after the change in the credit scoring function is turned off. This is shown in the right panel of Figure 9. Compared to the right panel of Figure 6, with the credit scoring function fixed, credit limits shrink for both groups but much more for the low credit score group. This result follows our previous discussion that a sizable share of credit extension effects arise from the endogenous response of the credit scoring function, which is more pronounced when the current credit score is lower.

The response of the credit scoring function also works as an amplification mechanism. As reported in Table 2, The aggregate consumption response with fixed credit scoring function is 19% (0.11% of 0.54%) lower measured on impact, or 22% (0.6% of 2.77%) lower measured by cumulated response. The endogenous response of credit scoring function amplifies the inter-temporal substitution effects: the change in the credit scoring function arises because both types of households start taking larger loans after the lowering of the interest rate, but the safe type does so *relatively* more.

Consumption responses in the model arise from the changes in wages and bond prices. Changes in bond prices result from changes in the risk-free real interest rate, repay probability function, and the credit scoring function (Equation (6)). To compare the effect through changes in the credit scoring function with other model mechanisms, I conduct counter-factual experiments by turning off responses of wages, the risk-free real interest rate, and the repay probability function, one at a time following similar procedures. The results are reported in Table 3.

	Δ C	Avg C	Δ Limit	Avg Limit
Benchmark	0.16%	0.54%	0.24%	0.66%
ψ	0.11%	0.44%	0.00%	0.33%
r	0.12%	0.43%	0.24%	0.58%
Pr repay	0.14%	0.52%	0.25%	0.46%
w	0.11%	0.14%	0.24%	0.66%

Table 3: Responses in Counter-factual Economies

Notes: These are responses on impact measured by percentage deviation. The first column describes which endogenous response is turned off. The second to the fifth columns describe the mean responses of model statistics or difference in responses between the highest and the lowest credit score groups. ΔC: difference in consumption responses. Avg C: mean consumption response. Δ Limit: difference in credit limit responses. Avg Limit: mean credit limit response.

As shown in the row "ψ", consistent with the discussion before, the response of the credit scoring function explains a sizable fraction of the difference in consumption responses and almost all difference in credit limit responses. As shown in row "r", turning off the response of the real interest rate lowers the average credit limit response but not the difference in credit limit responses. Also, it accounts for a smaller share of the difference in consumption responses. Contrary to the representative agent model, the aggregate consumption response to changes in real interest rate is rather weak. The amplification effect on aggregate consumption through response of the credit scoring function has equal quantitative importance (comparing Avg C in the row "ψ" and row "r"). As shown in row "Pr repay", the effects of the change in the repay probability function on credit limits and consumption are weak. Though limited enforcement is the origin for credit limits in the model, conditional on type the default probability changes little to the monetary policy shock and is not important for the transmission. It is indeed the change of composition of borrowers reflected by the change of credit scoring function that is responsible for the credit extension. We will return to this comparison when we discuss about the "risk-taking channel" in Section 6.4.

The effects of wage increase on consumption are strong, as shown in the row "w". Risky households are impatient and have higher marginal propensities to spend not only from extended credit but also from transitory income. They respond to the increased wage by spending more than the patient safe households. This explains a sizable fraction of the difference in consumption responses. Similar to Kaplan et al. (2015), the increase in wage also explains the majority of the aggregate consumption response[14]. However, by construction, the change in wage does not affect credit limits. This emphasizes the unique feature of credit *supply* mechanism: while the wage increases uniformly for all households and the induced heterogenous consumption responses arise from differences in *demand* responses, the response in credit supply through the credit scoring function by itself is different across households.

6.3 Heterogeneous Welfare Effects

What is the impact of monetary policy shocks on the welfare of households with different default risk and wealth levels? I construct the Consumption Equivalent Variation (CEV) measure for households within different wealth quintiles and credit score groups based on their ranks in the entire population at the stationary equilibrium[15].

	Wealth Quintile				
	$[0, 20\%)$	$[20\%, 40\%)$	$[40\%, 60\%)$	$[60\%, 80\%)$	$[80\%, 100\%)$
≤ 660	0.144%	0.122%	0.112%	0.105%	0.092%
$661 - 700$	0.102%	0.091%	0.087%	0.087%	0.087%
$701 - 740$	0.098%	0.083%	0.075%	0.072%	0.073%
> 740	0.092%	0.075%	0.064%	0.056%	0.050%

Table 4: CEV Conditional on Asset Quintile and Credit Score Group

Notes: The four credit score groups each corresponds to 28%, 16%, 19% and 37% of the population. Wealth quintile is based on the ranks of bond levels in the *entire* population at the stationary equilibrium.

As shown in Table 4, first, CEV is positive for households across all groups. This is because the monetary policy shock alleviates inefficiencies arising from monopolistic competitions. As for the heterogeneous effects, CEV is monotonically decreasing in both bond holding and credit score. In particular, households in the top wealth quintile and

[14]Heterogeneous marginal propensities to spend in Kaplan et al. (2015) arises from different composition of liquid and illiquid asset. The current paper with a different focus assumes heterogenous discount factors. The transmission mechanism through wage increase operates similarly.

[15]See C.3 for constructions of the CEV measure.

the highest credit score group on average experience 0.050% consumption-equivalent welfare gain while households in the bottom wealth quintile and the lowest credit score group on average experience 0.144% welfare gain. The monetary policy shock is more beneficial to the wealth poor than the wealth rich since it lowers the cost of borrowing as well as the return from saving[16]. It is more beneficial to households with lower credit score because through the credit channel, credit price improves more for the ex-ante more risky households.

6.4 Discussions on the Risk-Taking Channel

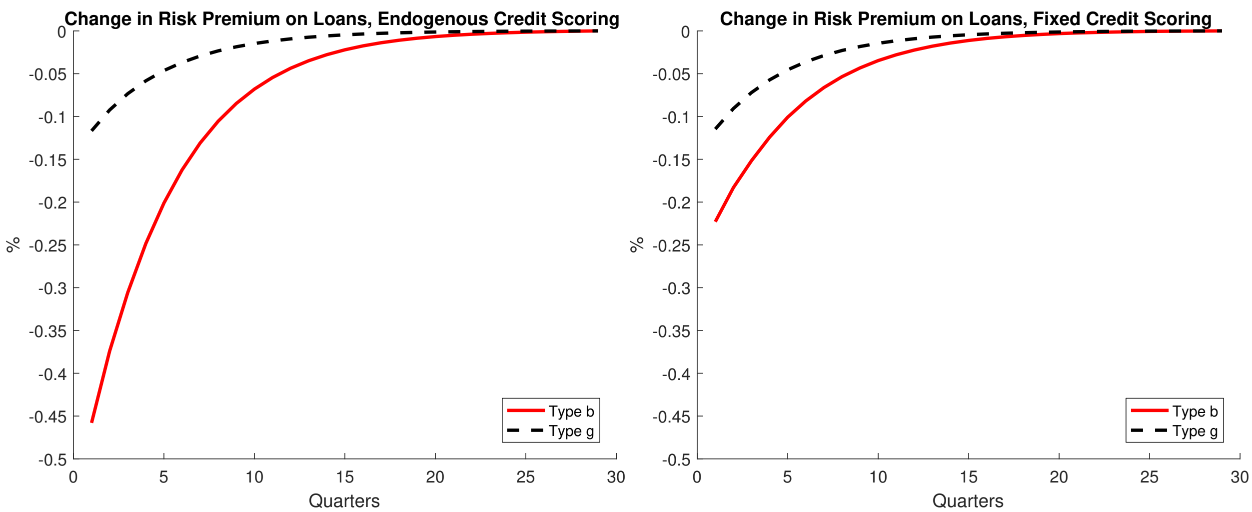

Figure 10: Responses of Risk Premium, Endogenous Belief v.s. Fixed Belief

Notes: These are change in absolute levels from those at the stationary equilibrium.

The traditional credit channel (Kiyotaki and Moore (1997), Bernanke et al. (1999)) focuses on the aggregate *volume* of credit, and monetary policy via the policy rate operates through the quantity but not the risk composition of credit (Collard et al. (2012)). Since the last financial crisis, there has been a growing literature studying the risk-taking behavior of financial intermediaries with or without the monetary policy context (Lorenzoni (2008), Bianchi (2011); Benigno et al. (2011), respectively). However, following the tradition, these papers exclusively interpret risk-taking as the quantity effect aka "overborrowing", rather than changes in "risk perception" or "risk assessment" as speculated by e.g. Borio and Zhu (2012), or the changes in risk composition of credit documented

[16]Asset is in the form of discount bonds in the model with the amount to be paid back or received contracted in the previous period. Therefore, the monetary policy shock does not affect the current payment or receipt but affects returns in future periods.

by recent empirical works Jiménez et al. (2012), Ioannidou et al. (2015), Dell'Ariccia et al. (2016), etc.

The current model predicts a risk-taking channel as monetary policy changes the risk assessment of lenders on borrowers. As shown in Figure 10, the left panel depicts the changes in average risk premium after the monetary policy shock for different risk groups of households. The average risk premium charged on loans taken by risky households drops by 45 basis point on impact in response to the 25 basis point negative nominal interest rate shock, while the drop in the average risk premium for safe households is much smaller. Notice that the changes in risk premium already account for the fact that both groups of households are taking larger loans after the shock, and that they would be charged with higher risk premium due to the larger loan sizes even if the policy rate were held constant.

The risk premium declines first because the cost of rolling over debt decreases and households are thus less prone to default. This is similar to the traditional credit channel in that credit limits arise from limited enforcement, and monetary policy changes the weighing between default and repayment. The difference along this dimension is that in traditional models a la Bernanke et al. (1999), the shock changes the cost of default (value of foregone collateral), while in the current model, the shock changes the cost of carrying debt while the cost of default is held relatively constant[17]. However, as discussed in Section 6.2, it is the change in the credit score (belief) updating function rather than the default probability function that accounts for the majority of the response in the bond price function. This is illustrated in the right panel of Figure 10: if the response in the credit scoring function were turned off, the difference in the risk premium responses between the two groups would not be as stark. These comparisons shed light on the risk assessment channel: given the type of borrowers, the default probability does not respond significantly to the monetary policy shock, but the safe type borrows relatively more and the pool of borrowers is perceived to be safer by creditors.

The model also implies that after the monetary policy shock, a larger fraction of aggregate credit is channeled to the more risky households. As shown by the dashed curve in the left panel of Figure 11, the fraction of credit lent to the risky households rises on impact of the shock and exhibits a hump shape response. This arises from both credit supply and demand effects. The credit price improves and credit limit is relaxed more for risky households, and they also borrow more out of the extended credit for being

[17] The fact that the value of reputation changes after the shock complicates such analysis. Indeed, after the shock, default behavior results in even a lower score since the safe households are even less likely to declare bankruptcy as the cost of carrying debts decreases.

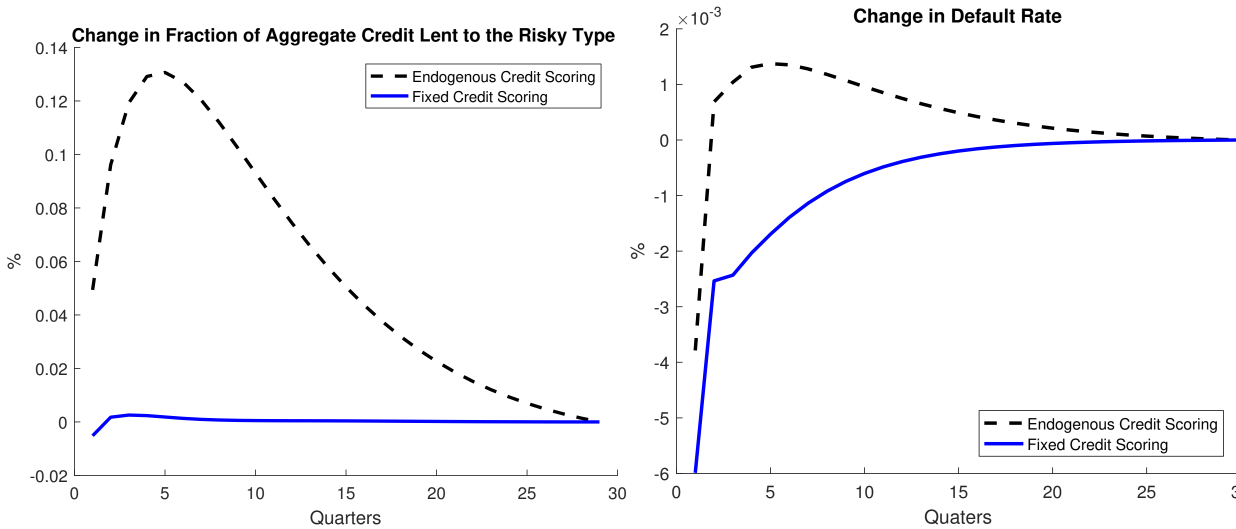

Figure 11: Change in the Risk Composition of Credit and Default Rate

Notes: These are change in absolute levels from those at the stationary equilibrium.

less patient. The average default rate exhibits overshooting during the transition path, as shown in the right panel of Figure 11. The default rate drops on impact but rises for a few quarters after the shock, precisely due to the increase in the fraction of debt held by risky households. It is worth emphasizing that the aggregate debt and the default rate conditional on risk type, together with other aggregate statistics, reverts monotonically to the stationary equilibrium. It is the change in the risk composition of credit that leads to the spike of default risk. This is illustrated by turning off the response in the credit score (belief) updating function, shown by the two solid curves in Figure 11. First of all, the response in the belief updating function is crucial in driving the change in the risk composition of credit, as shown in the left panel. Second, if the risk composition of credit were held unchanged, the default rate would revert back to the stationary equilibrium level monotonically, as shown in the right panel.

One may attempt to associate the analysis here with the episode prior to the 2008 financial crisis. After all, we had blamed the sub-prime loans led by the era of low interest rate for the cause of subsequent default crisis. I want to outline several caveats for such interpretation. First, though widely speculated, it is still an open question if the composition of credit had become more risky during this episode[18]. Second, for the current model to generate the change in the risk composition of credit, it is crucial to test whether the quality of pool of borrowers improves. This points to one unique feature of

[18] e.g. Foote et al. (2016) shows after accounting for the debt outflows of low-income borrowers, the allocations of debt across income groups remained stable.

the "risk-taking" channel in the current model in contrast to the bank-lending channel in e.g. Adrian and Shin (2010): the financial intermediaries in the current model are competitive and passive and the change in credit supply is a mere automatic response to the change in households' borrowing and default behaviors. The third caveat is by noticing that the change in default rate in the model is much smaller than that in the data[19]. Nevertheless, I am still optimistic that the current model can be used to address the mechanisms behind the crisis, with more empirical facts as backups and an active banking sector and mortgage debt to make the model more quantitative relevant.

7 Conclusions

The current paper documents a new fact that the consumption response is greater for households with higher default risk. Motivated by the fact, I study the endogenous response of credit supply to monetary policy shocks in a Heterogeneous Agent New Keynesian model augmented with information asymmetry. The key mechanism of the model lies in that financial intermediaries cannot observe households' risk types. Borrowing larger loans signals that the household is a more risky type and worsens the credit price as financial intermediaries factor in the additional default risk. The lower interest rate following an accommodative monetary policy shock encourages the less risky type to borrow and alleviates credit rationing due to adverse selection.

I show that the responses in credit supply arising from changes in lenders' beliefs account for a sizable share of the heterogeneous consumption responses and amplify the aggregate response.

While there is a large literature on credit channel and the credit rationing mechanism due to adverse selection dates back to Stiglitz and Weiss (1981), it is surprising little attention has been paid to incorporate this mechanism into a DSGE framework. The current paper is a first step toward this incorporation with a focus on the consumer side. The framework and ideas developed here can be carried over to the production side.

The current model focuses on the unsecured consumer loans and deliberately abstracts from collateral loans for tractability. The change in the default probability conditional on risk type is weak in the current setup which is helpful in highlighting the effects of adverse selection. It is worth exploring how monetary policy changes the price and value of collateral and has a stronger effect on the conditional default probability. It is more exciting to compare both channels and study their interactions. I leave these for

[19] e.g., the delinquency rate on single-family residential mortgages rises from 3.06% at 2007Q4 to 11.2% at 2010Q1, while the variation in the model default rate is less than 0.01%.

future research.

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Appendix A Empirical analysis

A.1 CEX sample treatment

Data from year 1996-2013 is acquired from the BLS website. Data before year 1996 is acquired from the ICPSR archive. Family characteristics are derived from *fmly* files. Expenditure is aggregated into categories (food, apparatus, dwelling, etc.) from the *mtab* file, based on the BLS official categorization of UCC (Universal Classification Code). The expenditure is first aggregated for each month and each household asked in an interview (recall households are asked to report the expenditure in the past 3 months), then aggregated within a quarter across households. Auto loan interest rate information is derived from *ovb* file, which is continuously provided since 1984Q1; therefore the samples in use are from 1984Q1 onward. Value of house, debt owed to creditors, amount of checking and saving accounts are derived from *itab* file by aggregating the corresponding UCC entries.

Expenditure is deflated using the category-specific CPI. For sample selection, I keep only households with head's age between 25 and 75. I keep only urban households. I remove households with 0 or negative food expenditure. I remove households with 0 or negative aggregate expenditure. The number of households in a month varies from 4128 to 9378 with an average of 5624 for the sample range 1984M1:2007M12.

A.2 Measuring households' default risk based on auto loan interest rate

Other things equal, households who are charged with higher interest rate on their auto loans are labeled as having higher default risk. I first estimate the following OLS[20]:

$$\begin{aligned} IR_i = \alpha_0 &+ \beta_1 \text{HouseholdsCharacteristics}_i + \beta_2 \text{LoanFeatures}_i \\ &+ \gamma_1 \text{PurchaseMonth}_i + \gamma_2 \text{InterviewMonth}_i + \varepsilon_i, \end{aligned} \tag{7}$$

where i indexes each auto loan. IR_i is the interest rate. Households' characteristics include age, family income, housing tenure status, and family size. Loan characteristics include the purchase price of vehicle, loan down payment ratio, and the type of vehicles.

[20] To save notations, the coefficients α,β etc. in each estimation equation should be treated as independent from other equations.

	All Households (1)	Has Auto Loan (2)	Has Default Risk (3)	Higher Risk (4)	Lower Risk (5)
# Observations	122495	47387	37509	18761	18748
Loan Interest Rate	-	0.10	0.10	0.13	0.08
Age	45.16	42.37	42.06	41.90	42.22
White	0.83	0.86	0.86	0.86	0.86
Renter	0.37	0.30	0.30	0.32	0.29
Non-mortgagor	0.19	0.13	0.13	0.13	0.13
Mortgagor	0.44	0.57	0.57	0.56	0.58
Family size	2.67	2.98	2.98	3.00	2.95
Edu year	13.33	13.63	13.66	13.58	13.73
After tax income	39357.83	48693.95	53744.26	52595.91	54893.40
Down payment ratio	-	0.1310	0.1307	0.1288	0.1325
Vehicle Price	-	13103.52	13203.44	12573.82	13833.50
Quarterly Expenditure	5688.88	7443.76	7538.95	7431.67	7644.79

Notes: Households are ranked based on the regression residual of the auto loan interest rate equation within each survey year. Households whose auto loan interest rate residuals lie above median are categorized as "Higher Default Risk". Households whose auto loan interest rate residuals lie below median are categorized as "Lower Default Risk".

Table A.1: Sample Mean Statistics

Table A.1 reports the mean statistics of household and loan characteristics for different samples. As shown in the first two columns, 47,387 out of 122,495, or 38.7% of total households, report having at least one auto loan. Compared to the average, households with auto loans are younger, more likely to be white, more likely to be a homeowner, have a larger family size, have slightly more education, and have higher income and expenditure. But overall, their characteristics are not very different from the average household. Therefore, though default risk can only be measured for a fraction of households which have reported auto loans, the consumption behavior of these households should represent the average household well. The average annualized auto loan interest rate is 10%. The down payment ratio of loan is around 13%. The average purchase price of vehicles is $13103.52 measured in the current price of each survey year.

Not all households report valid entries for the down payment ratio and the purchase price of vehicles. Table A.1 Column (3) reports the statistics for samples that have such information and are used in the baseline regression. Again, besides a slightly higher average income, they do not look very differently from the average household.

The estimation results for the loan interest rate equation specified in (7) are reported in Table A.2. Column 1 reports the baseline estimation based on which the default risk measure is constructed. Since some households report multiple auto loans, the total number of loan-level observations is 49,589. Overall, household and loan characteristics explain 32% of the variation in auto loan interest rates. In particular, households whose head is older, white, and more educated receive lower interest rates. Households with

	Auto Loan Interest Rate			
	(1)	(2)	(3)	(4)
HH Characteristics				
Age	-5.37e-05***	-7.73e-05***		-6.86e-05***
	(1.60e-05)	(1.43e-05)		(1.81e-05)
White	-0.00122**	-0.000217		-0.00218***
	(0.000523)	(0.000473)		(0.000586)
Non-mortgagor	-0.00225***	-0.00445***		-0.00243***
	(0.000669)	(0.000597)		(0.000750)
Mortgagor	-0.00131***	-0.00364***		-0.00183***
	(0.000458)	(0.000404)		(0.000529)
Edu year	-0.000511***	-0.000894***		-0.000382***
	(8.63e-05)	(7.54e-05)		(9.89e-05)
Family size	0.000489***	0.000951***		0.000551***
	(0.000126)	(0.000112)		(0.000144)
Log income	-3.74e-05	-0.00180***		-0.000349
	(0.000225)	(0.000197)		(0.000267)
Loan Characteristics				
Down payment ratio	0.00497***		0.00423***	0.00489***
	(0.00117)		(0.00109)	(0.00137)
Log purchase price	-0.0118***		-0.0125***	-0.0133***
	(0.000331)		(0.000260)	(0.000357)
Vehicle type dummies	Yes	Yes	Yes	Yes
Interview month dummies	Yes	Yes	Yes	Yes
Purchase month dummies	No	No	No	Yes
Observations	49,589	59,561	53,289	26,536
R-squared	0.317	0.287	0.313	0.290

Notes: "Non-mortgagor" and "Mortgagor" are relative to the base level "Renter". Only urban households and households with head aged between 25 and 75 are in sample. Robust standard errors are in parentheses. *** p<0.01, ** p<0.05, * p<0.1.

Table A.2: Determinants of Auto Loan Interest Rate larger family size receive higher interest rates. Home-owners receive lower interest rates compared to renters, and outright home-owners receive further lower interest rates compared to mortgagors. The effect of household income is not significant after controlling for the loan characteristics. Loans for vehicles with higher purchase prices and with lower down payment ratios carry lower interest rates.

With all these characteristics controlled, a sizable fraction of variation in auto loan interest rates is left unexplained. If lenders can observe additional risk characteristics of households such as the credit score and have factored the default risk into the loan interest rates, the unexplained auto loan interest rates should well capture households' default risk. The residual for each observation is thus used as a measure for the default risk of a loan. The average of residuals across all reported auto loans weighted by the purchase price of vehicles is used as a measure for the default risk of a household.

I rank households according to the default risk measure and split into two groups based on the ranks of the measure within each survey year. Column (4) and (5) in Table A.1 describe the mean statistics of the two groups. Since the default risk measure is constructed conditional on the observable characteristics, the distribution of observable characteristics do not differ significantly between the two groups. The average loan interest rate of the two groups are 13% and 8%, respectively.

Figure A.1 depicts the time series of quarterly expenditure for the two groups of households. After controlling for observable characteristics such as family income, the quarterly expenditure of the two groups tracks closely with each other for the estimation period. It is a known issue that the CEX under represents non-food expenditure in later waves (see e.g. Aguiar and Bils (2011)), so the total expenditure does not exhibit upward trending during the episode. However, since the current analysis focuses on the variation over the business cycle, there is hardly reason the measurement errors are correlated with monetary policy shocks and therefore the under-representation issue is less of a concern.

A.3 Identification with the local projection method

Following Jordà (2005) Coibion et al. (2012), and Ramey (2015), the local projection method linearly regresses the variables of interest on the monetary policy shocks and a set of controls. The specification is written as following:

$$X_{i,t+f} = \alpha^i_{0,f} + \beta^i_f S_t + B^i_f(L) X_{i,t-1} + C^i_f(L) S_{t-1} + D^i_f(L) Z_{t-1} + u_{i,t+f}, \tag{8}$$

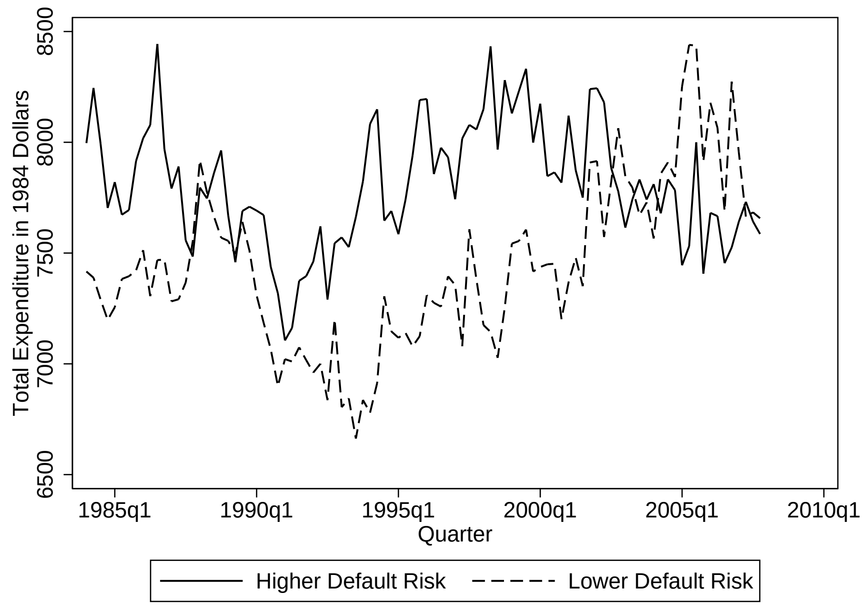

Figure A.1: Quarterly Expenditure in Sub-groups of households

where $i \in \{\text{less risky,more risky}\}$ is the group of households. f is the forward period ahead of time t. $X_{i,t+f}$ are the log of different measures of consumption for group i at time $t+f$. S_t is the policy surprise at time t. Z_{t-1} is a set of control variables including the funds rate itself, log CPI (seasonally adjusted), unemployment rate, and log industrial production (seasonally adjusted). (L) is a lag operator with the optimal lag periods chosen based on BIC criterion.

The coefficient for S_t estimated from an equation with forward period f can be interpreted as the effects of a one-period shock to federal funds rate on consumption at period f ahead. Collecting the coefficients varying the forward periods, I construct the consumption impulse responses for different groups of households, presented in Figure A.2.

As shown, the impulse responses generated by the local projection method has similar qualitative properties as generated by the SVAR, presented in Figure 1: (1) the aggregate responses measured by either the PCE account or the CES average have similar magnitude; (2) the consumption response for households with higher default risk is higher than the average while the consumption response for household with lower default risk is insignificant from zero. It is known that the local projection method produces more erratic impulse responses (Ramey (2015)), and the identified response is of greater magnitude (Coibion (2012)). The current analysis on consumption response exhibits similar properties.

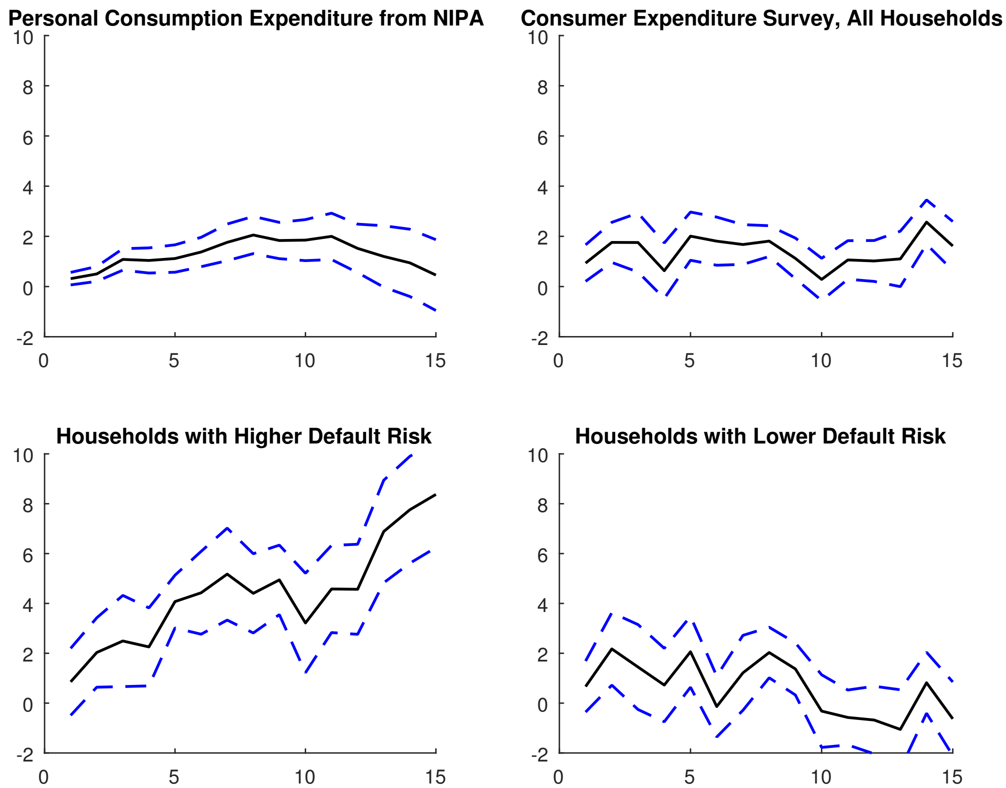

Figure A.2: Impulse responses of consumption, R&R shocks, local projection method

Notes: Units on the horizontal axis are quarters. Units on the vertical axis are percentage points. The solid lines are the impulse responses of consumption aggregate and of each subgroup of households to a 100 basis point negative innovation in federal funds rate. The dashed lines are one standard deviation intervals. Households' default risks are measured based on their auto loan interest rates. Monetary policy shocks are identified using R&R shocks with the local projection method. The data in use is from 1984Q1 to 2007Q4.

The local projection method allows to test the statistical significance of heterogeneous effects conveniently. For each forward period f, I run the linear regression pooling both risk groups and adding interactions between the group indicator and policy shocks:

$$\begin{aligned} X_{i,t+f} = \mathbb{1}(i=0)[\alpha^0_{0,f} + \beta^0_f S_t + B^0_f(L)X_{i,t-1} + C^0_f(L)S_{t-1} + D^0_f(L)Z_{t-1}] + \\ \mathbb{1}(i=1)[\alpha^1_{0,f} + \beta^1_f S_t + B^1_f(L)X_{i,t-1} + C^1_f(L)S_{t-1} + D^1_f(L)Z_{t-1}] + u_{i,t+f}, \end{aligned} \tag{9}$$

and the hypothesis $\beta^0_f = \beta^1_f$ tests whether there are heterogeneous effects for each forward period f.

Table A.3 shows the coefficients β^0_f and β^1_f estimated with Equation 9. As shown, the effects are insignificant from zero at most time horizons for households in the lower risk group. And the effects are significant for households in the higher risk group (note

Period	Lower Risk	Higher Risk	Difference	Period	Lower Risk	Higher Risk	Difference
1	-0.0094	-0.012	-0.0026	9	-0.0065	-0.035**	-0.0286
	(0.0114)	(0.011)	(0.0159)		(0.012)	(0.015)	(0.0192)
2	-0.0277**	-0.0128	0.0149	10	0.0051	-0.0256	-0.0307
	(0.0129)	(0.0133)	(0.0185)		(0.014)	(0.0178)	(0.0226)
3	-0.0128	-0.0225	-0.0098	11	0.0049	-0.0233	-0.0282
	(0.0137)	(0.0153)	(0.0205)		(0.0139)	(0.0172)	(0.0221)
4	-0.0074	-0.0167	-0.0093	12	0.0156	-0.0362*	-0.0518**
	(0.0125)	(0.014)	(0.0187)		(0.0138)	(0.0186)	(0.0231)
5	-0.0145	-0.0316**	-0.0171	13	0.0226	-0.0599***	-0.0824***
	(0.0116)	(0.0126)	(0.0171)		(0.0153)	(0.0204)	(0.0255)
6	0.0081	-0.0355**	-0.0436**	14	0.0027	-0.0601***	-0.0628***
	(0.0123)	(0.0147)	(0.0192)		(0.0132)	(0.0188)	(0.0229)
7	0.0014	-0.0448***	-0.0462**	15	0.0116	-0.0625***	-0.0741***
	(0.0162)	(0.0153)	(0.0222)		(0.0163)	(0.0179)	(0.0242)
8	-0.0065	-0.0443***	-0.0379*				
	(0.0124)	(0.0149)	(0.0194)				

Notes: This table shows the impulse responses of CEX-measured consumption to a 100 basis point monetary policy shocks, identified via the local projection method. Robust standard errors are in parentheses, corrected with the Newey-West method. *** p<0.01, ** p<0.05, * p<0.1.

Table A.3: Impulse Responses via the Local Projection Method

that the table presents the effects of a 100 basis point *rise* in the policy rate, so negative coefficients correspond to stimulating effects of accommodative monetary policy). The effects are greater for the higher risk group at all horizons, and most differences are statistically significant at the 10% level.

A.4 Identification using household-level variations

I have chosen the time-series identification methods based on grouped households as benchmark for two reasons. First, these methods are standard in the macroeconomic literature and the results can be compared to previous studies on aggregate dynamics. Second, since CEX only offers a short panel (up to 4 quarters) for each household, by grouping households within each quarter and forming these "synthetic panel series", I can analyze dynamics at a much longer horizon. The major concern in using the grouping method is that households may select in and out of groups in response to macroeconomic policy shocks. The identified consumption responses may not come from change in consumption behaviors but purely come from change in the group compositions.

To address this issue, I directly use the panel feature of the data and identify the effects of monetary policy shocks using household-level variations. Specifically, I estimate the following equation:

$$X_{it} = \beta_0 + \beta_1 S_t + \beta_2 Risk_i \cdot S_t + \gamma Z_t + \eta_i + \varepsilon_{it}, \tag{10}$$

where i indexes household and t indexes time. X_{it} is the quarterly expenditure, S_t is the monetary policy shock (cumulative R&R shock), Z_t are the aggregate control variables, and η_i is the household dummy to control for fixed effects. $Risk_i$ is the measure of default risk for each household which does not vary over time (since the vehicle information is surveyed only once for each household). I measure default risk using both the loan interest rate residuals directly and the ranks of residuals within each survey year.

	Log Quarterly Expenditure				
	(1)	(2)	(3)	(4)	(5)
R&R	-0.00725*	-0.00875	0.00987	0.00480	0.0247
	(0.00382)	(0.00819)	(0.0107)	(0.0284)	(0.0301)
R&R × Residual		-0.303*			
		(0.183)			
R&R × Risk group			-0.0390***	-0.0389***	
			(0.0150)	(0.0150)	
R&R × Risk percentile					-0.000760***
					(0.000252)
R&R × Age				0.000406	0.000379
				(0.000640)	(0.000639)
R&R × Non-mortgagor				-0.0321	-0.0313
				(0.0266)	(0.0265)
R&R × Mortgagor				-0.0131	-0.0130
				(0.0163)	(0.0163)
Observations	264,697	77,168	77,168	77,168	77,168
Number of Households	109,641	33,876	33,876	33,876	33,876

Notes: R&R is the cumulative Romer and Romer shock. "Residual" is constructed using the regression residual from the auto loan interest rate equation. "Risk percentile" is constructed by ranking the regression residual within each survey year. "Risk group" splits households into two equal-sized groups based on the ranks. All regressions control for household fixed effects, quarter fixed effects, the interaction between risk measure and quarter fixed effects, time trend, the interaction between risk measure and time trend, and aggregate variables (unemployment and log industrial production). Standard errors in parentheses are robust and clustered at the household level. *** p<0.01, ** p<0.05, * p<0.1.

Table A.4: Heterogeneous Effects of Monetary Policy on Household Consumption

Table A.4 reports the estimation results. All regressions control for household fixed effects, quarter fixed effect, time trend, and aggregate variables. I use the cumulative Romer and Romer shock as monetary policy shock, denoted by the variable "R&R". Column (1) shows that in response to a 100 basis point positive innovation in federal funds rate, the average quarterly consumption decreases by 0.7%. This effect is of similar magnitude as identified using the SVAR or local projection method with aggregate time series data, reported in the previous section. Column (2) uses the auto loan interest rate regression residual as the measure for default risk. The coefficient for the interaction between R&R and the residual is negative and significant at 10% level , showing that the effect of monetary policy on consumption is greater for households with higher default risk.

A better measure of default risk is the rank of the regression residual within a survey year instead of the absolute level. "Risk group" in Column (3) is a indicator that assigns 1 to households whose default risks lie above the median and assigns 0 otherwise (this is exactly the same indicator based on which households are split into two risk groups to construct the time series variables in the previous section). The coefficient for the interaction between R&R and the risk group indicator is negative and significant at 1% level, showing again that the effect of monetary policy is greater for households in the more risky group. The magnitude in the difference is large. While the effect on consumption for the less risky group is insignificantly from zero, the effect for the more risky group is around 3%.

One concern is that the measure of default risk may be correlated with other household characteristics. The regression reported in Column (4) controls for the interaction between R&R shock and the age of household head, and the interaction between R&R shock and housing tenure. The effect remains to be significantly different between the two risk groups, and the magnitude remains to be large. Though the literature has found heterogenous effects of monetary policy for households across age groups (Wong (2015)) and housing tenure groups (Cloyne et al. (2015)), I do not find significant results after controlling for households' default risk as shown in Column (4). Finally, in Column (5) I use the percentile of interest rate regression residual as the measure for default risk. The interaction between R&R shock and this measure is also significantly negative.

A.5 Imputing propensities of loan delinquency from SCF

The CEX data does not survey loan repayment information. However, the rich demographics and financial information contained allows one to impute loan repayment behavior from the SCF data. In particular from all SCF surveys, households are asked the following question: *"...were all payments made the way they were scheduled during the last year, or were payments on any of the loans sometimes made later or missed?"*. I flag households as delinquent if they answer yes to the question.

I first estimate a probit equation to regress the delinquent flag on a set of observable characteristics using the SCF:

$$E[D_i|\text{Demographics}_i, \text{FinancialInfo}_i] = F(\alpha_0 + \beta_1\text{Demographics}_i + \beta_2\text{FinancialInfo}_i), \tag{11}$$

where D_i is the delinquent flag. F is the cumulated distribution function for standard normal distribution. Households' demographics information includes age, race, years of education, marital status, employment status, pre-tax family income, and family size. Financial information includes whether having checking or saving account, unsecured debt to income ratio, whether a home owner, whether has mortgage if a homeowner, rent payment if not a home owner, and the mortgage interest coverage ratio if having mortgages. These variables are selected to reflect all potential factors that may affect households' loan repayment behavior and have counterparts in the CEX.

A separate equation 11 is estimated for each wave of the SCF. Then the estimated coefficients are used to predict the probability of being delinquent using the corresponding variables from the CEX for each household. Since the SCF is surveyed less frequently, for years that the CEX is surveyed but the SCF is not, the estimated coefficients for the nearest year are used for the imputation. The R-square statistics for each imputation equation is around 10% to 20%. Correlation between the measure of default risk by auto loan interest rate and the imputed delinquency probability is around 0.1 and significant for all waves.

Figure A.3 depict the impulse responses constructed by SVAR and the local projection method for the two groups of households divided based on the imputed default risk. As shown, for both pairs of impulse responses, the consumption response for households with higher default risk is greater than households with lower default risk. Thus the results are robust to different measures of default risk.

It is worth discussion that two measures can be viewed as capturing different parts of the default risk. In particular, the default risk measured by the auto loan interest rate is the "residual" default risk conditional on households' observable characteristics, and is closer to the model concept of "credit score". Instead, the imputed delinquency propensity from the SCF is a summary statistics of households' observable characteristics. But if one assumes not all characteristics used in the imputation equation are observable to the lenders (e.g. the family size, rent payment, etc.), then the imputed delinquency also captures some portion of the "residual" default risk. This explains why the overall correlation between the two measures is not extremely high, but the results of heterogeneous

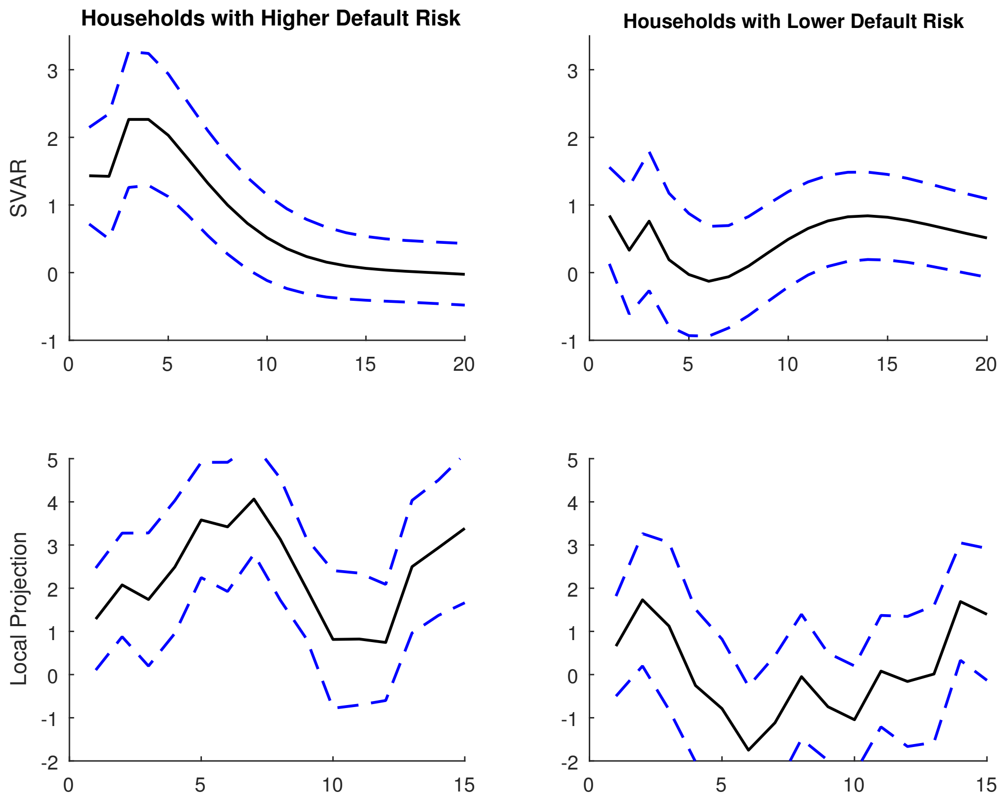

Figure A.3: Impulse responses of consumption, Risk Imputed from SCF

Notes: Units on the horizontal axis are quarters. Units on the vertical axis are percentage points. The solid lines are the impulse responses of consumption aggregate and of each subgroup of households to a 100 basis point negative innovation in federal funds rate. The dashed lines are one standard deviation intervals. Households' default risks are imputed delinquency rate from the SCF. The upper two graphs depict impulse responses generated by SVAR. The lower two graphs are generated by local projection method.

responses hold consistently with the two different measures.

A.6 Heterogenous responses within subgroups

One concern is that the measure of default risk is correlated with other households' characteristics, and heterogeneous consumption responses may be due to differences in other households' characteristics rather than default risk[21]. Therefore, for robustness check, I ask if heterogeneous consumption responses across risk groups persist within each subgroup.

[21]Indeed, the two measures of default risk have accounted for observable characteristics. But since they are based on parametric estimations, such issue is still possible.

A.6.1 Housing tenure groups

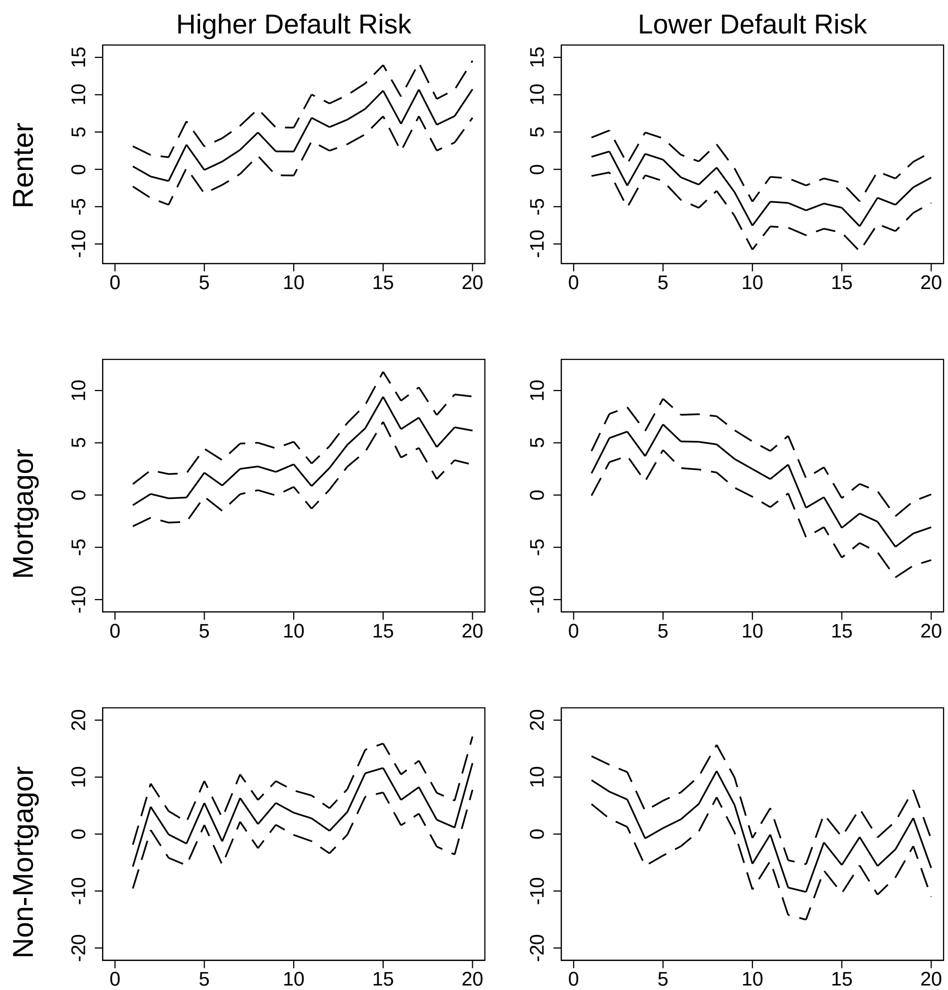

Figure A.4: Impulse responses of consumption by home tenure and risk groups

Notes: Units on the horizontal axis are quarters. Units on the vertical axis are percentage points. The solid lines are the impulse responses of consumption to a 100 basis point negative innovation in federal funds rate. The dashed lines are one standard deviation intervals. Households' default risk is measured based on risk premium charged on their auto loan interest rates.

Cloyne et al. (2015) have shown consumption impulse response to monetary policy shocks is larger for households with mortgage debt. I divide households from CEX into three groups: renters, homeowners with mortgage, and homeowners without mortgage. Then within each housing tenure group and survey year, I split households into two groups based on the default risk measured by auto loan interest rates (the rank is within each household tenure group and survey year). I then average consumption across households within each housing tenure and risk group for each quarter. I run a separate set of regressions for each subgroup following the local projection method specified in Appendix A.3.

The consumption impulse responses for each housing tenure and risk group are depicted in Figure A.4. As shown, for renters, the consumption impulse response peaks at 10% for the more risky group, and is in significant from 0 for the less risky group. For households with mortgage debt, similar patterns hold but the differences in responses between the two risk groups are of smaller magnitude. For homeowners without mortgage debt, the average response is greater, but the differences between the two risk groups are no longer significant. The differences in consumption responses are significant for renters and become less significant for homeowners. This can be explained by that renters are more likely to face credit constraints arising from credit risk, while homeowners are able to resort to collateral for borrowing.

A.6.2 Age groups

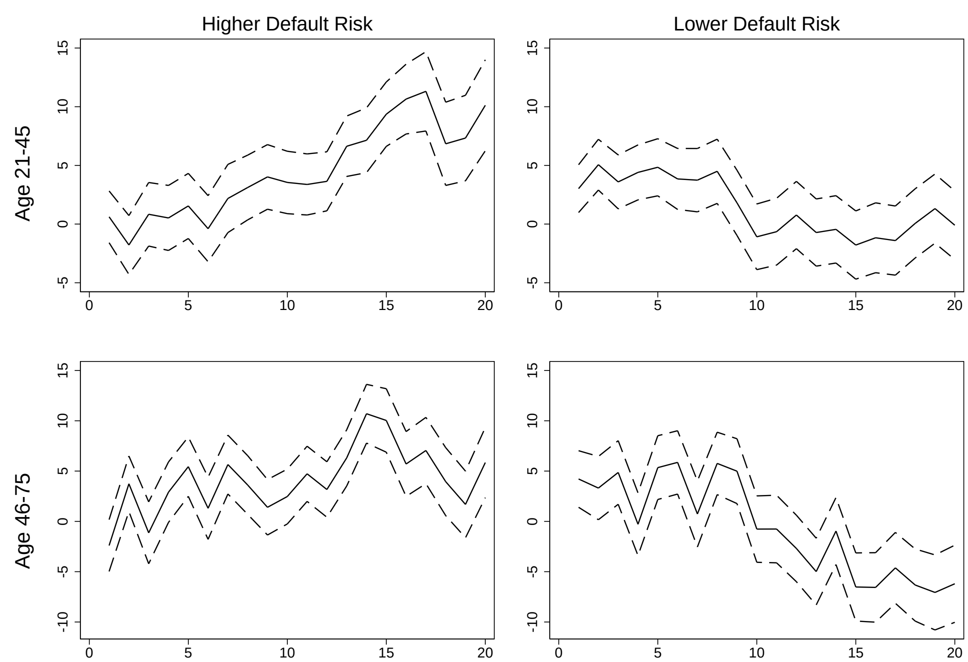

Figure A.5: Impulse responses of consumption by age and risk groups

Notes: Units on the horizontal axis are quarters. Units on the vertical axis are percentage points. The solid lines are the impulse responses of consumption to a 100 basis point negative innovation in federal funds rate. The dashed lines are one standard deviation intervals. Households' default risk is measured based on risk premium charged on their auto loan interest rates.

Wong (2015) has documented that consumption elasticity to change in interest rate is greater for younger age group. I divide households to two groups with age 21-45 and age 46-75. Within each age and survey year group, I then split households into two risk groups based on the default risk measured by auto loan interest rates (again households are ranked within each age group and survey year). Figure A.5 depicts the impulse responses for the 4 groups. As shown, within both age group, the consumption response is greater for households with higher default risk. And the differences are larger for the younger age group.

Appendix B Proofs

B.1 Existence proof

The proof for existence of stationary equilibrium $SCE(r, Tr, \tau)$ proceeds in the following three steps:

- Step 1: fixing (r, Tr, τ, w), I define $\mathbb{T}$ maps from the set of bond price, credit score, and (transformed) value function $(q, \psi, \mathscr{W})$ to itself. I prove $\exists \sigma_\varepsilon^*$, s.t. $\forall \sigma_\varepsilon > \sigma_\varepsilon^*$, $\mathbb{T}$ has fixed points. I eventually use the Schauder fixed point theorem. The challenge involved is to show $(q, \psi, \mathscr{W})$ are Lipschitz continuous in s. The is solved by noticing that the mapping $\mathbb{T}$ amplifies the Lipschitz conditions of $(q, \psi, \mathscr{W})$, but to a factor that goes to 0 as σ_ε goes to infinity. This is established in Lemma 10, 14, and 15. Then I prove the mapping $\mathbb{T}$ itself and the induced policy function are continuous in $(q, \psi, \mathscr{W})$. This is established through Lemma 20, 21, 22, 23, 24. Then the existence of fixed points of $\mathbb{T}$ follows Schauder fixed point theorem (Lemma 25).

- Step 2: fixing $(q, \psi, \mathscr{W})$ and the induced policy function, I construct a mapping $\mathbb{T}_4$ that maps the space of probability measure to itself. I show that $\mathbb{T}_4$ is continuous with weak topology in Lemma 26. Similar proof strategies have been used in Cao (2016). The space of probability measure is compact because the state space is the product space of finite sets and a closed subset of $[0, 1]$. Then the existence of fixed point of $\mathbb{T}_4$ follows again from Schauder fixed point theorem (Lemma 27).

- Step 3: I construct remaining equilibrium objects from $(q, \psi, \mathscr{W}, \Phi)$ in Lemma 28. There are two key features of the model why such construction can be done. (1) The steady state wage w is completely determined by two model parameters: the productivity level and the elasticity of substitution in final goods production function. (2) Government bond is used to absorb any excess saving or borrowing, and government spending is used as a residual to balance the government budget. Therefore, there is no price feedback from the probability measure Φ.

I first revisit and state formally the domain and range of each function at the stationary equilibrium. The observable states are denoted by $x = (e, a, s) \in \mathbb{X} = \mathbb{E} \times \mathbb{A} \times \mathbb{S}$. The set for unobserved type is denoted by $\mathbb{I} = \{b, g\}$. The bond price function denoted by $q(a', x) : \mathbb{A} \times \mathbb{X} \to [0, \frac{1}{1+r}]$. The posterior function denoted by $\xi^{(d,a')}(x) : \mathbb{Y} \times \mathbb{X} \to [0,1]$. The credit scoring function denoted by $\psi^{(d,a')}(x) : \mathbb{Y} \times \mathbb{X} \to \mathbb{S} = [\underline{s}, \bar{s}] \subsetneq [0,1]$. The value function denoted by $W(i,x) : \mathbb{I} \times \mathbb{X} \to \mathbb{R}$. The policy function denoted by $m^{(d,a')}(i,x) : \mathbb{Y} \times \mathbb{I} \times \mathbb{X} \to [0,1]$. Denote the space for function h as $\mathbb{B}^h$. Later on as I proceed with the proofs, I endow $\mathbb{B}^h$ with more properties.

Lemma 2. *Fix any $r > 0, w > 0, \tau \in [0,1), Tr \geq 0$, $q \in \mathbb{B}^q$, and $\psi \in \mathbb{B}^\psi$, the value function $W(i,x)$ and policy function $m^{(d,a')}(i,x)$ exist and satisfy*

$$W(i,x) = \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log\Big(\sum_{(d,a') \in \mathbb{M}(x)} \exp\big(\frac{U^{(d,a')}(i,x) + \beta_i \tilde{W}^{(d,a')}(i,x)}{\sigma_\varepsilon} \big)\Big), \tag{12}$$

$$m^{(d,a')}(i,x) = \begin{cases} \dfrac{\exp\big(\frac{U^{(d,a')}(i,x)+\beta_i \tilde{W}^{(d,a')}(i,x)}{\sigma_\varepsilon}\big)}{\sum_{(\tilde{d},\tilde{a}') \in \mathbb{M}(x)} \exp\big(\frac{U^{(\tilde{d},\tilde{a}')}(i,x)+\beta_i \tilde{W}^{(\tilde{d},\tilde{a}')}(i,x)}{\sigma_\varepsilon}\big)}, & \text{for } (d,a') \in \mathbb{M}(x), \\ 0, & \text{for } (d,a') \notin \mathbb{M}(x), \end{cases} \tag{13}$$

where

$$\tilde{W}^{(d,a')}(i,x) \triangleq \sum_{i' \in \{b,g\}} \sum_{e' \in \mathbb{E}} \Omega(i'|i) \Gamma(e'|e) W(i', e', a', \psi^{(d,a')}(x)),$$

σ_ε is the scale parameter of Type-I extreme value distribution, and γ^C is the Euler's constant.

Further more, W is bounded by constants $\underline{W}$ and $\overline{W}$ that are irrelevant of choices of q and ψ, i.e., fix any $r > 0, w > 0, \tau \in [0,1)$, and $Tr \geq 0$, $\exists \underline{W}(r,w,\tau,Tr)$ and $\overline{W}(r,w,\tau,Tr)$ s.t. $\forall q \in \mathbb{B}^q$ and $\psi \in \mathbb{B}^\psi$, $\underline{W}(r,w,\tau,Tr) \leq W(i,x) \leq \overline{W}(r,w,\tau,Tr), \forall i, x.$

Proof. Consider the mapping $\tilde{\mathbb{T}}: \mathbb{B}^V \to \mathbb{B}^V$ defined as following:

$$\tilde{\mathbb{T}}W(i,x) = E[\max_{(d,a')\in\mathbb{M}(f,e,a,s)} U^{(d,a')}(i,x) + \varepsilon^{(d,a')} \\ + \beta_i \sum_{i'\in\{b,g\}} \Omega(i'|i) \sum_{e'} \Gamma(e'|e)\ W(i',e',a',\psi^{(d,a')}(x))],$$

where the expectation operator is w.r.t the Type-I extreme value (T1EV) shock $\boldsymbol{\varepsilon}$. By property of T1EV shock, the mapping is equal to

$$\tilde{\mathbb{T}}W(i,x) = \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log\Big(\sum_{(d,a')\in\mathbb{M}(x)} \exp\Big(\frac{U^{(d,a')}(i,x) + \beta_i \tilde{W}^{(d,a')}(i,x)}{\sigma_\varepsilon}\Big)\Big).$$

And the policy function is given by (13).

In Part 1, I prove $\tilde{\mathbb{T}}$ preserves uniform boundedness. Indeed, I prove a stronger equiboundedness condition which is independent of the choice of q and ψ. Formally, fix any $r > 0, w > 0, \tau \in [0,1),$ *and* $Tr \geq 0$, $\exists \overline{W}$ and $\underline{W}$ s.t. $\forall q \in \mathbb{B}^q$ and $\psi \in \mathbb{B}^\psi$, if $\underline{W} \leq W(i,x) \leq \overline{W}, \forall i,x$, then $\underline{W} \leq \tilde{\mathbb{T}}W(i,x) \leq \overline{W}, \forall i,x$. This is basically because default choice is always feasible, and flow utility is bounded by the finite set of asset choices and boundedness of bond price function.

First $\underline{W} \leq W(i,x) \leq \overline{W} \Rightarrow \underline{W} \leq \tilde{W}^{(d,a')}(i,x) \leq \overline{W}$, since $\tilde{W}$ is convex combination of W. Consider $c^{(1,0)}(x) = wen(e)(1-\tau) + Tr > 0$, therefore default choice is always feasible and the flow utility upon default is regardless of q or ψ, denote $\underline{U} \triangleq \min\{U^{(1,0)}(b,e_{min},a,s), U^{(1,0)}(g,e_{min},a,s)\}$, then

$$\sum_{(d,a')\in\mathbb{M}(x)} \exp\Big(\frac{U^{(d,a')}(i,x) + \beta_i \tilde{W}^{(d,a')}(i,x)}{\sigma_\varepsilon}\Big) \geq \exp\Big(\frac{\underline{U} + \beta_i \tilde{W}^{(1,0)}(i,x)}{\sigma_\varepsilon}\Big) \\ \geq \exp\Big(\frac{\underline{U}}{\sigma_\varepsilon}\Big) \cdot \exp(\beta_i \underline{W}/\sigma_\varepsilon).$$

Therefore,

$$\tilde{\mathbb{T}}W(i,x) \geq \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log\Big(\exp\Big(\frac{\underline{U}}{\sigma_\varepsilon}\Big) \cdot \exp(\beta_i \underline{W}/\sigma_\varepsilon)\Big) \\ \geq \sigma_\varepsilon \gamma^C + \underline{U} + \beta_i \underline{W}.$$

Therefore, it suffices to impose $\sigma_\varepsilon \gamma^C + \underline{U} + \beta_i \underline{W} \geq \underline{W}$ to have $\tilde{\mathbb{T}}W(i,x) \geq \underline{W}$. Therefore choose $\underline{W} = \min_{i\in\{b,g\}} \dfrac{\sigma_\varepsilon \gamma^C + \underline{U}}{1-\beta_i}$, we have $\tilde{\mathbb{T}}W(i,x) \geq \underline{W}$.

Next consider $\forall (d,a')$ and x, $c^{(d,a')}(x) \leq \bar{c} \triangleq we_{max} n(e_{max})(1-\tau) + Tr + a_{max} -$

$\frac{1}{1+r}a_{min}$. Denote $\overline{U} = u(\bar{c} - v(n(e_{max})))$, then

$$\sum_{(d,a')\in\mathbb{M}(x)} \exp(\frac{U^{(d,a')}(i,x) + \beta_i \tilde{W}^{(d,a')}(i,x)}{\sigma_\varepsilon}) \leq (N_{\mathbb{A}}+1)\exp(\frac{\overline{U}}{\sigma_\varepsilon}) \cdot \exp(\beta_i \overline{W}/\sigma_\varepsilon).$$

Therefore,

$$\begin{aligned}
\tilde{\mathbb{T}}W(i,x) &\leq \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log((N_{\mathbb{A}}+1)\cdot \exp(\frac{\overline{U}}{\sigma_\varepsilon}) \cdot \exp(\beta_i \overline{W}/\sigma_\varepsilon)) \\
&\leq \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log(N_{\mathbb{A}}+1) + \overline{U} + \beta_i \overline{W},
\end{aligned}$$

where $N_{\mathbb{A}}$ is the number of elements in finite set $\mathbb{A}$.

Choose $\overline{W} = \max_{i\in\{b,g\}} \frac{\sigma_\varepsilon \gamma^C + \overline{U} + \sigma_\varepsilon \log(N_{\mathbb{A}}+1)}{1-\beta_i}$, we have $\tilde{\mathbb{T}}W(i,x) \leq \overline{W}$.

In Part 2, I verify $\tilde{\mathbb{T}}$ satisfies Blackwell's sufficient conditions. Monotonicity trivially holds. Now consider $\forall v_0 \geq 0$,

$$\tilde{\mathbb{T}}(W+v_0)(i,x) = \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log(\sum_{(d,a')\in\mathbb{M}(x)} \exp(\frac{U^{(d,a')}(i,x) + \beta_i(\tilde{W}^{(d,a')}(i,x)+v_0)}{\sigma_\varepsilon})),$$

which uses the fact that $\tilde{W}$ is convex combination of W. Therefore,

$$\begin{aligned}
\tilde{\mathbb{T}}(W+v_0)(i,x) &= \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log(\exp(\frac{\beta_i v_0}{\sigma_\varepsilon}) \sum_{(d,a')\in\mathbb{M}(x)} \exp(\frac{U^{(d,a')}(i,x)\beta_i \tilde{W}^{(d,a')}(i,x)}{\sigma_\varepsilon})) \\
&= \sigma_\varepsilon \gamma^C + \sigma_\varepsilon \log(\sum_{(d,a')\in\mathbb{M}(x)} \exp(\frac{U^{(d,a')}(i,x) + \beta_i(\tilde{W}^{(d,a')}(i,x)+v_0)}{\sigma_\varepsilon})) + \beta_i v_0 \\
&= \tilde{\mathbb{T}}W(i,x) + \beta_i v_0.
\end{aligned}$$

Therefore, discounting holds with module β_g.

Using the Blackwell's sufficient conditions for contraction mapping (Stokey et al. (1989), Theorem 3.3), working on the space $\mathbb{B}^V$ for value function uniformly bounded by $\underline{W}$ and $\overline{W}$ established in Part 1, I arrive at the existence of value function given by (12). □

Remaining existence proofs are in Online Appendix D.

To facilitate the exposition of model properties and the numerical algorithm, it is convenient to write down some equilibrium conditions here.

$$odd^{(d,a')}(x) = \begin{cases} \frac{\exp(\beta_b \frac{\tilde{W}^{(d,a')}(b,x)}{\sigma_\varepsilon})}{\exp(\beta_g \frac{\tilde{W}^{(d,a')}(g,x)}{\sigma_\varepsilon})} \frac{\sum_{(\tilde{d},\tilde{a}') \in \mathbb{M}(x)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(i,x)}{\sigma_\varepsilon} + \frac{\tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}') \in \mathbb{M}(x)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(i,x)}{\sigma_\varepsilon} + \frac{\tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon}))}, \text{ for } (d,a') = (0,a') \\ \frac{\exp(\frac{U^{(1,0)}(b,x)}{\sigma_\varepsilon} + \beta_b \frac{\tilde{W}^{(1,0)}(b,x)}{\sigma_\varepsilon})}{\exp(\frac{U^{(1,0)}(g,x)}{\sigma_\varepsilon} + \beta_g \frac{\tilde{W}^{(1,0)}(g,x)}{\sigma_\varepsilon})} \frac{\sum_{(\tilde{d},\tilde{a}') \in \mathbb{M}(x)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(i,x)}{\sigma_\varepsilon} + \frac{\tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}') \in \mathbb{M}(x)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(i,x)}{\sigma_\varepsilon} + \frac{\tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon}))}, \text{ for } (d,a') = (1,0) \end{cases} \tag{14}$$

$$\xi^{(d,a')}(x) = \frac{1}{1 + \frac{1-s}{s} \cdot odd^{(d,a')}(x)} \tag{15}$$

$$\psi^{(d,a')}(x) = \xi^{(d,a')}(x)\Omega(g|g) + (1 - \xi^{(d,a')}(x))\Omega(g|b); \tag{16}$$

$$\begin{aligned} q(a', x) = & \frac{1}{1+r} \sum_{e' \in \mathbb{E}} \{s[1 - m^{(1,0)}(g, e', a', \psi^{(0,a')}(e, a, s))] \\ & + (1-s)[1 - m^{(1,0)}(b, e', a', \psi^{(0,a')}(e, a, s))]\}; \end{aligned} \tag{17}$$

$$\begin{aligned} \Phi(i', e', a', S') = & \int \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) \sum_{a'} m^{(0,a')}(i, x)\mathbb{1}(\psi^{(0,a')}(x) \in S')\Phi(i, e, a, ds) \\ & + \mathbb{1}(a' = 0) \int \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) m^{(1,0)}(i, x)\mathbb{1}(\psi^{(1,0)}(x) \in S')\Phi(i, e, a, ds) \end{aligned} \tag{18}$$

B.2 Proofs for model properties

Proof for Lemma 1:

Proof. Given $\{P_t, w_t\}$, an intermediate goods firm solves

$$\begin{aligned} J_t(p_{t-1}) = \max_{p_t, y_t, n_t} y_t \frac{p_t}{P_t} - w_t n_t - \frac{\theta}{2}(\frac{p_t}{p_{t-1}} - \bar{\pi})^2 Y_t + \frac{1}{1 + r_{t+1}} J_{t+1}(p_t), \\ s.t. \quad y_t = z n_t, y_t = (\frac{p_t}{P_t})^{-\eta} Y_t. \end{aligned}$$

Denote $m_t = \frac{w_t}{z}$ the marginal cost for production, we simplify the above to

$$J_t(p_{t-1}) = [(\frac{p_t}{P_t} - m_t)(\frac{p_t}{P_t})^{-\varepsilon} - \frac{\theta}{2}(\frac{p_t}{p_{t-1}} - \bar{\pi})^2]Y_t + \frac{1}{1 + r_{t+1}} J_{t+1}(p_t),$$

and the necessary condition for optimization is:

$$\text{FOC:} \quad [\frac{1}{P_t}(\frac{p_t}{P_t})^{-\varepsilon} + (\frac{p_t}{P_t} - m_t)(-\varepsilon)(\frac{p_t}{P_t})^{-\varepsilon} p_t^{-1} - \frac{\theta}{p_{t-1}}(\frac{p_t}{p_{t-1}} - \bar{\pi})]Y_t + \frac{1}{1+r_{t+1}} J'_{t+1}(p_t) = 0 \tag{19}$$

$$\text{Envelope Theorem: } J'_t(p_{t-1}) = \theta(\frac{p_t}{p_{t-1}} - \bar{\pi})\frac{p_t}{p_{t-1}^2} Y_t \tag{20}$$

Plug 20 into 19, we have

$$[\frac{1}{P_t}(\frac{p_t}{P_t})^{-\varepsilon} + (\frac{p_t}{P_t} - m_t)(-\varepsilon)(\frac{p_t}{P_t})^{-\varepsilon} p_t^{-1} - \frac{\theta}{p_{t-1}}(\frac{p_t}{p_{t-1}} - \bar{\pi})]Y_t + \frac{1}{1+r_{t+1}}\theta(\frac{p_{t+1}}{p_t} - \bar{\pi})\frac{p_{t+1}}{p_t^2} Y_{t+1} = 0.$$

Impose symmetry s.t. $p_t = P_t$ we have

$$[1 + (1 - m_t)(-\varepsilon) - \theta\frac{P_t}{P_{t-1}}(\frac{P_t}{P_{t-1}} - \bar{\pi})]Y_t + \frac{1}{1+r_{t+1}}\theta(\frac{P_{t+1}}{P_t} - \bar{\pi})\frac{P_{t+1}}{P_t} Y_{t+1} = 0,$$

i.e.,

$$[\varepsilon(m_t - m^*) - \theta\pi_t(\pi_t - \bar{\pi})]Y_t + \frac{1}{1+r_{t+1}}\theta\pi_{t+1}(\pi_{t+1} - \bar{\pi})Y_{t+1} = 0.$$

Rearrange we have

$$\pi_t(\pi_t - \bar{\pi}) = \frac{\varepsilon}{\theta}(m_t - m^*) + \frac{1}{1+r_{t+1}}(\pi_{t+1} - \bar{\pi})\pi_{t+1}\frac{Y_{t+1}}{Y_t},$$

where $m^* = \frac{\varepsilon-1}{\varepsilon}$. □

Lemma 3. *Given wage w_t and tax rate τ_t, the labor supply is a function of labor efficiency e only.*

Proof. Consider the static problem,

$$\max_{c,n} c - v(n) \quad s.t. \quad c = (1-\tau_t)w_t e n + b_t,$$

where b_t is the budget accounting for current and future bonds and transfer. From first order condition:

$$v'(n) = (1-\tau_t)w_t e.$$

Since $v'(n) > 0, v''(n) > 0, v'(0) = 0, v'(\infty) = \infty$, inverting $v'(n)$ we have

$$n_t(e) = v'^{-1}((1-\tau_t)w_t e).$$

$\square$

Assumption 1 assumes the type is not persistent, so in equilibrium credit scoring function is irrelevant of actions. This assumption eventually assumes both types share the same continuation values and thus simplifies analysis.

Lemma 4. *With Assumption 1, in any stationary equilibrium, the credit scoring function satisfies* $\psi^{(d,a')}(e,a,s) = 1/2, \forall (d,a') \in \mathbb{M}(e,a,s), e, a, s.$

Proof. By definition of ψ

$$\psi^{(d,a')}(e,a,s) = \xi^{(d,a')}(x)\Omega(g|g) + (1-\xi^{(d,a')}(x))\Omega(g|b) = 1/2,$$

when $\Omega_{g|g} = \Omega_{g|b} = 1/2$. $\square$

With this result that choices do not affect future credit scores, I can show the value function is increasing in bond holding.

Lemma 5. *With Assumption 1, in any stationary equilibrium, the value function satisfies* $W(i,e,\tilde{\tilde{a}},s) \leq W(i,e,\tilde{a},s), \forall \tilde{\tilde{a}} \leq \tilde{a}, \forall i,e,s.$

Proof. According to Lemma 4, with Assumption 1, $\psi \equiv 1/2$ at stationary equilibrium. Therefore, the continuation value defined in Lemma 2 satisfies

$$\begin{aligned}\tilde{W}^{(d,a')}(i,x) &= \sum_{i' \in \{b,g\}} \sum_{e' \in \mathbb{E}} \Omega(i'|i)\Gamma(e'|e)W(i',e',a',\psi^{(d,a')}(x)) \\ &= \sum_{i' \in \{b,g\}} \sum_{e' \in \mathbb{E}} \frac{1}{2}\Gamma(e'|e)W(i',e',a',1/2).\end{aligned}$$

Therefore $\tilde{W}^{(d,a')}(i,x)$ does not depend on (i,a,s). Define

$$\hat{W}(e,a') \triangleq \tilde{W}^{(0,a')}(i,e,a,s) = \sum_{i' \in \{b,g\}} \sum_{e' \in \mathbb{E}} \frac{1}{2}\Gamma(e'|e)W(i',e',a',1/2), \tag{21}$$

$$\hat{W}^D(e) \triangleq \tilde{W}^{(1,0)}(i,e,a,s) = \sum_{i' \in \{b,g\}} \sum_{e' \in \mathbb{E}} \frac{1}{2}\Gamma(e'|e)W(i',e',0,1/2). \tag{22}$$

Using Lemma 2, the Bellman equation for W can be written as

$$W(i,x) = \sigma_\varepsilon\gamma + \sigma_\varepsilon \log\Big(\sum_{(0,a')\in\mathbb{M}(x)} \exp\Big(\frac{U^{(0,a')}(i,x) + \beta_i \hat{W}(e,a')}{\sigma_\varepsilon}\Big) + \exp\Big(\frac{U^{(1,0)}(i,x) + \beta_i \hat{W}^D(e)}{\sigma_\varepsilon}\Big)\Big).$$

Fixing (i,e,s) and $\tilde{\tilde{a}} \le \tilde{a}$, since $\mathbb{M}(e,\tilde{\tilde{a}},s) \subseteq \mathbb{M}(e,\tilde{a},s)$ and $U^{(0,a')}(i,e,\tilde{\tilde{a}},s) \le U^{(0,a')}(i,e,\tilde{a},s), \forall (0,a') \in \mathbb{M}(e,\tilde{\tilde{a}},s)$, we have $W(i,e,\tilde{\tilde{a}},s) \le W(i,e,\tilde{a},s)$. □

Proof for Proposition 2:

Proof. Consider the set of transitory preference shocks that the default choice is optimal

$$BR(i,e,a,s) \triangleq \{\boldsymbol{\varepsilon} : U^{(1,0)}(i,e,a,s) + \beta_i \hat{W}^D(e) + \varepsilon^{(1,0)} > U^{(0,a')}(i,e,a,s) + \beta_i \hat{W}(e,a') + \varepsilon^{(0,a')},$$
$$\forall (0,a') \in \mathbb{M}(e,a,s)\}. \tag{23}$$

Fixing (i,e,s) and $\tilde{\tilde{a}} \le \tilde{a}$, since $U^{(1,0)}(i,e,\tilde{\tilde{a}},s) = U^{(1,0)}(i,e,\tilde{a},s)$, $\mathbb{M}(e,\tilde{\tilde{a}},s) \subseteq \mathbb{M}(e,\tilde{a},s)$, and $U^{(0,a')}(i,e,\tilde{\tilde{a}},s) \le U^{(0,a')}(i,e,\tilde{a},s), \forall (0,a') \in \mathbb{M}(e,\tilde{\tilde{a}},s)$, therefore $BR(i,e,\tilde{a},s) \subseteq BR(i,e,\tilde{\tilde{a}},s)$. Therefore $m^{(1,0)}(i,e,\tilde{\tilde{a}},s) \ge m^{(0,1)}(i,e,\tilde{a},s)$.

By definition of q,

$$q(a',x) = \frac{1}{1+r} \sum_{e'\in\mathbb{E}} \{s[1 - m^{(1,0)}(g,e',a',\psi^{(0,a')}(e,a,s))]$$
$$+ (1-s)[1 - m^{(1,0)}(b,e',a',\psi^{(0,a')}(e,a,s))]\}.$$

From Lemma 4, $\psi \equiv 1/2$. From first part of the proof $\forall \tilde{\tilde{a}}' \le \tilde{a}' < 0$

$$m^{(1,0)}(i',e',\tilde{\tilde{a}}',1/2) \ge m^{(1,0)}(i',e',\tilde{a}',1/2).$$

Therefore,

$$q(\tilde{\tilde{a}}',x) \le q(\tilde{a}',x).$$

□

Proof for Proposition 3:

Proof. According to Equation 15, for $d = 0$,

$$\xi^{(d,a')}(x) = \frac{1}{1 + \frac{1-s}{s} \cdot \frac{\exp(\beta_b \frac{\tilde{W}^{(d,a')}(b,x)}{\sigma_\varepsilon}) \sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x)+\beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\exp(\beta_g \frac{\tilde{W}^{(d,a')}(g,x)}{\sigma_\varepsilon}) \sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x)+\beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon})}}.$$

Therefore, to show $\xi^{(0,\tilde{\tilde{a}}')}(x) \le \xi^{(0,\tilde{a}')}(x)$, it suffices to prove

$$\frac{\exp(\beta_b \tilde{W}^{(0,\tilde{\tilde{a}}')}(b,x))}{\exp(\beta_g \tilde{W}^{(0,\tilde{\tilde{a}}')}(g,x))} \ge \frac{\exp(\beta_b \tilde{W}^{(0,\tilde{a}')}(b,x))}{\exp(\beta_g \tilde{W}^{(0,\tilde{a}')}(g,x))}.$$

With Assumption 1 and the definition of $\hat{W}$ (Equation 21) in the proof for Lemma 5, $\tilde{W}^{(0,a')}(g,x) = \tilde{W}^{(0,a')}(b,x) = \hat{W}(e,a')$. Therefore it suffices to prove

$$\frac{\exp(\beta_b \hat{W}(e,\tilde{\tilde{a}}'))}{\exp(\beta_g \hat{W}(e,\tilde{\tilde{a}}'))} \ge \frac{\exp(\beta_b \hat{W}(e,\tilde{a}'))}{\exp(\beta_g \hat{W}(e,\tilde{a}'))}$$
$$\Leftrightarrow (\beta_g - \beta_b)(\hat{W}(e,\tilde{a}') - \hat{W}(e,\tilde{\tilde{a}}')) \ge 0.$$

From Lemma 5, $\hat{W}(e,\tilde{a}') \ge \hat{W}(e,\tilde{\tilde{a}}')$; and $\beta_g > \beta_b$ holds by assumption. □

Proof for Proposition 4:

Proof. Recall the definition of default set in Equation 23. To prove Proposition 4, it suffices to prove $\exists \phi^*$ s.t. $\forall \phi_g < \phi^*$, $BR(g,e,a,s;\phi_g) \subseteq BR(b,e,a,s;\phi_g)$. Since $U^{(0,a')}(b,e,a,s;\phi_g) = U^{(0,a')}(g,e,a,s;\phi_g)$, it suffices to show $\forall(e,a,s,a')$

$$U^{(1,0)}(g,e,a,s;\phi_g) + \beta_g(\hat{W}^D(e;\phi_g) - \hat{W}(e,a';\phi_g)) \le U^{(1,0)}(b,e,a,s;\phi_g) + \beta_b(\hat{W}^D(e;\phi_g) - \hat{W}(e,a';\phi_g)),$$

or,

$$U^{(1,0)}(g,x;\phi_g) - U^{(1,0)}(b,x;\phi_g) \le (\beta_g - \beta_b)[\hat{W}(e,a';\phi_g) - \hat{W}^D(e;\phi_g)].$$

First, $\hat{W}^D(e;\phi_g) < \bar{W}$, where $\bar{W}$ is constructed in Lemma 2. Notice $\bar{W}$ is a function of (r,τ,Tr) but does not depend on ϕ_g, since the maximum period utility is attained with highest consumption leisure bundle without default discount. Therefore, it suffices to prove

$$U^{(1,0)}(g,x;\phi_g) - U^{(1,0)}(b,x;\phi_g) \le (\beta_g - \beta_b)\hat{W}(e,a';\phi_g) - (\beta_g - \beta_b)\bar{W}.$$

Now consider

$$\hat{W}(e,a';\phi_g) = \sum_{i'\in\{b,g\}}\sum_{e'\in\mathbb{E}}\frac{1}{2}\Gamma(e'|e)W(i',e',a',1/2;\phi_g)$$
$$\geq \frac{1}{2}W(b,e_{min},a',1/2;\phi_g) + \frac{1}{2}W(g,e_{min},a',1/2;\phi_g).$$

And

$$W(i,e_{min},a',1/2;\phi_g) \geq \sigma_\varepsilon\gamma^C + U^{(1,0)}(i,x;\phi_g) + \beta_i[\frac{1}{2}W(b,e_{min},0,1/2;\phi_g) + \frac{1}{2}W(g,e_{min},0,1/2;\phi_g)].$$

This uses the fact that default is always an option and debt is cleared after default. But,

$$W(i,e_{min},0,1/2;\phi_g) \geq \underline{W}',$$

where $\underline{W}'$ does not depend on ϕ_g, since with 0 bond holding, continue always choosing 0 bond in all future periods is always an option, and the flow utility of choosing 0 bond does not depend on ϕ_g. Therefore,

$$W(i,e_{min},a',1/2;\phi_g) \geq \sigma_\varepsilon\gamma + U^{(1,0)}(i,x;\phi_g) + \beta_i\underline{W}',$$

and it thus suffices to prove

$$U^{(1,0)}(g,x;\phi_g) - U^{(1,0)}(b,x;\phi_g) \leq (\beta_g-\beta_b)\frac{1}{2}(U^{(1,0)}(g,x;\phi_g) + U^{(1,0)}(b,x;\phi_g)) + \sigma_\varepsilon\gamma^C$$
$$+\frac{1}{2}(\beta_g+\beta_b)\underline{W}' - (\beta_g-\beta_b)\bar{W},$$

or, equivalently,

$$(1-\frac{1}{2}(\beta_g-\beta_b))U^{(1,0)}(g,x;\phi_g) \leq (1+\frac{1}{2}(\beta_g-\beta_b))U^{(1,0)}(b,x;\phi_g) + W_0, \tag{24}$$

where $W_0 = \sigma_\varepsilon\gamma^C + \frac{1}{2}(\beta_g+\beta_b)\underline{W}' - (\beta_g-\beta_b)\bar{W}$ is a constant that depends on (r,τ,Tr) but not ϕ_g. Recall,

$$U^{(1,0)}(i,x;\phi_g) = u(\phi_i[c-v(n(e))]),\ \text{where}\ c = (1-\tau)wen(e) + Tr.$$

Therefore $U^{(1,0)}(b,x;\phi_g)$ does not depend on ϕ_g. Since $u(\cdot)$ satisfies the inada condition and the right-hand-side of Equation 24 does not depend on ϕ_g, $\exists\phi^*$ s.t. $\forall\phi_g < \phi^*$, Equation 24 holds. □

Proof for Proposition 5:

Proof. Formally households facing the one-period bond price function q solves the following problem

$$\max_{(d,a')} U^{(d,a')}(i,x;q) + \beta_i \tilde{W}^{(d,a')}(i,x) + \varepsilon^{(d,a')},$$

where $\tilde{W}$ is the continuation value at the stationary equilibrium price function.

Now consider the posterior probability for non-default choices:

$$\xi^{(0,a')}(e,a,s;q) = \frac{s}{s + (1-s)\frac{\exp(\beta_b \frac{\tilde{W}^{(d,a')}(b,x)}{\sigma_\varepsilon})}{\exp(\beta_g \frac{\tilde{W}^{(d,a')}(g,x)}{\sigma_\varepsilon})} \frac{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;q)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;q)+\beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;q)} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;q)+\beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon})}}.$$

Consider $\forall \tilde{q}, \tilde{\tilde{q}}$, s.t. $\tilde{\tilde{q}} = (1+\Delta)\tilde{q}$, $\Delta > 0$. Since $\xi^{(0,a')}(e,a,s;\tilde{q}) > 0$ and $\xi^{(0,a')}(e,a,s;\tilde{\tilde{q}}) > 0$, to show $\xi^{(0,a')}(e,a,s;\tilde{\tilde{q}}) - \xi^{(0,a')}(e,a,s;\tilde{q}) \geq 0$, it suffices to show

$$\begin{aligned}
&\frac{\xi^{(0,a')}(e,a,s;\tilde{\tilde{q}})}{\xi^{(0,a')}(e,a,s;\tilde{q})} \geq 1 \\
\Leftrightarrow\ & \frac{\exp(\beta_b \frac{\tilde{W}^{(d,a')}(b,x)}{\sigma_\varepsilon})}{\exp(\beta_g \frac{\tilde{W}^{(d,a')}(g,x)}{\sigma_\varepsilon})} \frac{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{\tilde{q}})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{\tilde{q}})+\beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{\tilde{q}})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{\tilde{q}})+\beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon})} \\
\leq\ & \frac{\exp(\beta_b \frac{\tilde{W}^{(d,a')}(b,x)}{\sigma_\varepsilon})}{\exp(\beta_g \frac{\tilde{W}^{(d,a')}(g,x)}{\sigma_\varepsilon})} \frac{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{q})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{q})+\beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{q})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{q})+\beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon})} \\
\Leftrightarrow\ & \frac{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{\tilde{q}})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{\tilde{q}})+\beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{\tilde{q}})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{\tilde{q}})+\beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon})} \leq \frac{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{q})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{q})+\beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon})}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{q})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{q})+\beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon})} \\
\Leftrightarrow\ & \sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{\tilde{q}})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{\tilde{q}}) + \beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon}) \sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{q})} \exp(\frac{U^{(\tilde{d},\tilde{a})}(x;\tilde{q}) + \beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon}) \\
\leq\ & \sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{q})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{q}) + \beta_g \tilde{W}^{(\tilde{d},\tilde{a}')}(g,x)}{\sigma_\varepsilon}) \sum_{(\tilde{d},\tilde{a}')\in\mathbb{M}(x;\tilde{\tilde{q}})} \exp(\frac{U^{(\tilde{d},\tilde{a}')}(x;\tilde{\tilde{q}}) + \beta_b \tilde{W}^{(\tilde{d},\tilde{a}')}(b,x)}{\sigma_\varepsilon}).
\end{aligned} \tag{25}$$

Notice the flow utility from non default choices does not depend on i. And under Assumption 1, we use the definition of continuation value $\tilde{W}$ in Equation 21. Now we prove the following: $\forall \tilde{a}', \tilde{\tilde{a}}'$,

$$\begin{aligned}
&\exp(\frac{U^{(0,\tilde{a}')}(e,a;\tilde{q}) + \beta_g \hat{W}(e,\tilde{a}')}{\sigma_\varepsilon}) \cdot \exp(\frac{U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{\tilde{q}}) + \beta_b \hat{W}(e,\tilde{\tilde{a}}')}{\sigma_\varepsilon}) \\
&+ \exp(\frac{U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{q}) + \beta_g \hat{W}(e,\tilde{\tilde{a}}')}{\sigma_\varepsilon}) \cdot \exp(\frac{U^{(0,\tilde{a}')}(e,a;\tilde{\tilde{q}}) + \beta_b \hat{W}(e,\tilde{a}')}{\sigma_\varepsilon}) \\
&\geq \exp(\frac{U^{(0,\tilde{a}')}(e,a;\tilde{\tilde{q}}) + \beta_g \hat{W}(e,\tilde{a}')}{\sigma_\varepsilon}) \cdot \exp(\frac{U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{q}) + \beta_b \hat{W}(e,\tilde{\tilde{a}}')}{\sigma_\varepsilon}) \\
&+ \exp(\frac{U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{\tilde{q}}) + \beta_g \hat{W}(e,\tilde{\tilde{a}}')}{\sigma_\varepsilon}) \cdot \exp(\frac{U^{(0,\tilde{a}')}(e,a;\tilde{q}) + \beta_b \hat{W}(e,\tilde{a}')}{\sigma_\varepsilon}).
\end{aligned} \tag{26}$$

After rearranging, the above is equivalent to

$$\begin{aligned}
&\exp(\beta_b \hat{W}(e,\tilde{\tilde{a}}') + \beta_b \hat{W}(e,\tilde{a}'))[\exp((\beta_g - \beta_b)\hat{W}(e,\tilde{a}')) - \exp((\beta_g - \beta_b)\hat{W}(e,\tilde{\tilde{a}}'))] \\
&\cdot [\exp(U^{(0,\tilde{a}')}(e,a;\tilde{q}) + U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{\tilde{q}})) - \exp(U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{q}) + U^{(0,\tilde{a}')}(e,a;\tilde{\tilde{q}}))] \geq 0.
\end{aligned} \tag{27}$$

For $\tilde{a}' \leq \tilde{\tilde{a}}'$, by Lemma 5, $\hat{W}(e,\tilde{a}') \leq \hat{W}(e,\tilde{\tilde{a}}')$. Now observe

$$U^{(0,a')}(e,a;q) = u[a + (1-\tau)wen(e) + Tr - q(a')a' - v(n(e))].$$

Since $\tilde{\tilde{q}} = (1+\Delta)\tilde{q}$ and $\Delta > 0$, we have

$$U^{(0,\tilde{a}')}(e,a;\tilde{q}) - U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{q}) \leq U^{(0,\tilde{a}')}(e,a;\tilde{\tilde{q}}) - U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{\tilde{q}}).$$

Therefore we have

$$\exp(U^{(0,\tilde{a}')}(e,a;\tilde{q}) + U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{\tilde{q}})) - \exp(U^{(0,\tilde{\tilde{a}}')}(e,a;\tilde{q}) + U^{(0,\tilde{a}')}(e,a;\tilde{\tilde{q}})) \leq 0$$

and (27) holds. Symmetrically, (27) holds for $\tilde{a}' \geq \tilde{\tilde{a}}'$.

(27) implies (26). And expand (25) both sides we can see it is implied by (26).

□

Appendix C Quantitative Analysis

C.1 Solving Stationary Equilibrium and Calibration

The model partly features explicit aggregation in the sense that the policy function - the probability over discrete choices - has analytical expressions as in Equation 13. I utilize this feature by using a mixture of stochastic and non-stochastic simulation methods. To solve the stationary equilibrium where aggregate conditions are involved, I use the non-stochastic method introduced by Young (2010). To compute statistics for each credit score group, I use the Monte-Carlo simulation method.

C.1.1 Solving stationary equilibrium

I approximate the value, policy, bond pricing, and credit scoring function over a predetermined finite grid on the credit score s (notice all other state variables are discrete). For evaluation of these functions off grids, I use linear interpolation. Following the discussion in Hatchondo et al. (2010), instead of solving the bond price function and value function using nested loops, I solve them jointly. Indeed, I use a single loop to solve the value function, bond price function, credit scoring function, distribution, and aggregate prices jointly. The algorithm is presented as follows:

1. Guess aggregate price and quantities (B, Tr, G), value function W, bond price function q, credit scoring function ψ, and distribution Φ. Φ is approximated by the histogram on the discrete grids of credit score s (and other discrete state variables).

2. Solve households' Bellman equation according to the analytical expression in Equation 13, using the guessed W as next period value function, and the guessed q, ψ, Tr. For future credit score generated by ψ that is not on grid, I approximate future value using linear interpolation. The solution gives policy function m_new and value function W_new.

3. Compute the implied credit scoring function ψ_new consistent with the solved policy function m_new. Notice instead of using the policy function directly, Equation 15 should be used so that the flow utility of the two types can be canceled out for non-default choices, leaving ψ_new well-defined for all discrete choices (even for actions that are not feasible).

4. Compute the implied bond pricing function consistent with the solved policy function m_new and risk-free interest rate r, according to Equation 17.

5. Iterate forward Φ once to get Φ_new using the implied policy function m_new and credit scoring function ψ_new, according to Equation 18. This is done following the non-stochastic simulation proposed by Young (2010), in which the transition for discrete state variable is directly computed using the policy function m_new and exogenous transition matrix for labor efficiency and type-switching shocks,

while the transition for the continuous state variable generated by ψ_new is approximated by assigning mass to the adjacent grid points proportionally whenever the continuous variable is off grid. The procedure can be also interpreted as solving an approximating equilibrium defined in Chatterjee et al. (2011).

6. Compute implied prices and quantities $(w\_new, B\_new, Tr\_new, G\_new)$ using the implied distribution Φ_new.

7. Using a line search to update the initial guess. Update variable X according to $X = speed \cdot X\_new + (1 - speed) \cdot X$, where $speed$ is the step size used in line search, and X are the equilibrium objects specified before. Repeat step 1-7 until X is close to X_new under certain criteria.

It should be clear that the above procedure exactly follows the recursive mapping defined in B.1. For the detailed numerical implementations, the discrete set for bond values is chosen to be 61 equally spaced points from -1.5 to 3, which corresponds to $-\%150$ and %300 of the median annual labor earnings. The discrete grid points for credit score is $\{0.1, 0.5, 0.9, 0.95, 0.9633, 0.9766, 0.99\}$. More grid points are put on the higher end to capture the property that the calibrated transition matrix for type-switching shocks generates larger mass on higher credits cores at the stationary equilibrium. The above choices imply a total of 61(bond) $\times$ 7(credit score) $\times$ 7(labor) $\times$ 2(type) $=$ 5978 states, and $5978/2 \times 61 = 182329$ total entries in the pricing or credit scoring function. In the benchmark calibration, a step size $speed = 0.1$ ensures convergence. The distance between X and X_new is solved to $1e - 8$ under the sup-norm metric within 5000 iterations.

C.1.2 Simulation

Since the model is taken to match the cross-sectional distribution moments by credit score groups, I use a Monte-Carlo simulation method to construct these moments after solving the stationary equilibrium. I start with N agents with initial bond holding 0, initial credit score generated from a uniform distribution over the ergodic interval, and the initial labor efficiency level and risk type generated from their corresponding invariant distribution. I simulate the economy forward for T periods using the solved transition rules (policy function and credit scoring function). Then I construct the cross sectional moments in the final period.

T is chosen large enough so that aggregate and cross sectional moments do not vary significantly, indicating stationary distribution. In practice, I choose $T = 1000$ and $N = 4e5$.

C.1.3 Fitting to data using SMM

The parameters to be calibrated are $\Lambda = (\chi, z, \beta_g, \beta_b, \phi_g, \phi_b, \Omega_{gg}, \Omega_{bb}, \sigma_\varepsilon)$. After solving each stationary equilibrium and simulation, based on the simulated samples I construct the three aggregate moments: average hours, median wage, and government bond to output ratio; and the 12 cross-sectional moments: the credit limits, default rate, and marginal propensity of spending out of extended credit for the 4 credit score groups. Then I construct the following L^2 metric between model and data:

$$Metric(\Lambda) = \sum_{i=1}^{15} (\frac{M_i^{model} - M_i^{data}}{M_i^{data}})^2,$$

that is, I measure the distance between model and data using the relative distance and equal weights. I search parameters Λ to minimize the above criterion using a global optimization routine (genetic algorithm).

C.2 Solving Transitional Equilibrium

During the transitional path, government transfer and government bond are held constant, and the government spending is used as a residual to balance the budget. The following steps outline the shooting algorithm:

1. Fix the transition period T, Guess a sequence of aggregate price and quantities $\{Y_t, G_t, r_t, i_t, \pi_t\}$, value function $\{W_t\}$, bond price function $\{q_t\}$, credit scoring function $\{\psi_t\}$, and distribution $\{\Phi_t\}$ (again distribution is approximated by the histogram over discrete grids of credit score and other discrete state variables).

2. For every period t, I take the guessed $G_t, r_t, i_t, \pi_t, W_{t+1}, q_t, \psi_t, \Phi_t$ as given, but solve the wage w_t together with the policy function m_new_t to clear the goods market and government budget. This is in the spirit of Krusell and Smith (1997) that in the iterative procedure, the inter-temporal prices are fixed but the with-in period price is solved to clear the intra-period market. It turns out this is a necessary procedure to guarantee the iterative algorithm outlined here converges robustly. (Instead of solving backwards with guessed wage w_t and updating w_t afterwards.)

3. Compute the implied W_new_t, q_new_t, and ψ_new_t consistent with the solved policy m_new_t. Iterate Φ_t from $t = 1$ forward to get the implied distribution Φ_new_t. Compute aggregate Y_new_t based on distribution Φ_new_t and w_t. Compute π_new_t that solves the Non-linear Phillips curve. Compute i_new_t given by

the Taylor rule. Compute r_new_t given by the Fisher equation.

4. Update equilibrium objects with a line search: $X_t = speed \cdot X\_new_t + (1 - speed) \cdot X_t$. Repeat Step 1-4 until X_t and X_new_t are close enough.

In the benchmark experiment with the Taylor rule shocks, I set $T = 50$, and a step size of 0.01 ensures convergence, with distance between X_t and X_new_t smaller than $1e - 4$ under sup-norm metric after 1000 iterations.

C.3 Welfare Changes Measured by CEV

The Consumption Equivalent Variation (CEV) $\lambda(i, e, a, s)$ for consumer with state variable (i, e, a, s) is defined as satisfying the following:

$$E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u((1+\lambda)c_t - v(n_t), d_t) + \varepsilon^{(d_t, a'_t)}] = E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u(\tilde{c}_t - v(\tilde{n}_t), \tilde{d}_t) + \varepsilon^{(\tilde{d}_t, \tilde{a}'_t)}],$$

where $\{c_t, n_t, (d_t, a'_t)\}$ are the choices generated by policy functions for state variable (i, e, a, s) at the stationary equilibrium, and $\{\tilde{c}_t, \tilde{n}_t, (\tilde{d}_t, \tilde{a}'_t)\}$ are generated by policy functions after the monetary policy shock hits. Note by definition:

$$W(i, e, a, s) = E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u(c_t - v(n_t), d_t) + \varepsilon^{(d_t, a'_t)}],$$
$$\tilde{W}_1(i, e, a, s) = E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u(\tilde{c}_t - v(\tilde{n}_t), \tilde{d}_t) + \varepsilon^{(\tilde{d}_t, \tilde{a}'_t)}],$$

where $W, \tilde{W}_1$ are the value functions integrated over the transitory preference shocks at the stationary equilibrium and with the monetary policy shock, respectively. By rearranging:

$$\tilde{W}_1(i, e, a, s) = E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u((1+\lambda)c_t - v(n_t), d_t) + \varepsilon^{(d_t, a'_t)}] =$$
$$W(i, e, a, s) + E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u((1+\lambda)c_t - v(n_t), d_t)] - E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u(c_t - v(n_t), d_t)].$$

Therefore, $\lambda(i, e, a, s)$ can be solved as satisfying the following:

$$E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u((1+\lambda)c_t - v(n_t), d_t)] - E \sum_{t=1}^{\infty} (\Pi_{t'=1}^{t-1} \beta_{i_{t'}})[u(c_t - v(n_t), d_t)]$$
$$= \tilde{W}_1(i, e, a, s) - W(i, e, a, s). \tag{28}$$

The above procedure essentially avoids the difficulty in computing $E\sum_{t=1}^{\infty}(\Pi_{t'=1}^{t-1}\beta_{i_{t'}})\varepsilon^{(d_t,a'_t)}$, because in the simulation one does not evaluate $\varepsilon^{(d_t,a'_t)}$ directly (but only the probability over choices).

Equation (28) can be solved using a simulate method as following. For fixed λ, starting from a distribution of N agents with state (i,e,a,s), simulate forward for T periods, then approximate $E\sum_{t=1}^{\infty}(\Pi_{t'=1}^{t-1}\beta_{i_{t'}})[u((1+\lambda)c_t - v(n_t), d_t)]$ using the sum for the truncated T periods and computing the empirical mean across agents. Then search for λ to solve Equation (28). I have chosen $N = 4e5$ and $T = 50$ in the quantitative analysis. Similarly, instead of for agent with specific (i,e,a,s), CEV can be computed conditional on agents with certain statistics such as asset and credit score ranks as reported in Table 4.

Online Appendix I

Appendix D Additional Existence Proofs

Denote $\mathbb{B}^q = \{q : \mathbb{A} \times \mathbb{E} \times \mathbb{A} \times \mathbb{S} \to [0, \frac{1}{1+r}]\}$, $\mathbb{B}^\psi = \{\psi : \mathbb{Y} \times \mathbb{E} \times \mathbb{A} \times \mathbb{S} \to \mathbb{S}\}$, $\mathbb{B}^{\mathscr{W}} = \{\mathscr{W} : \mathbb{I} \times \mathbb{E} \times \mathbb{A} \times \mathbb{S} \to \mathbb{R}\}$.

I slightly abuse the notation by defining $\mathbb{B}^q(\mathscr{L}_q)$ the subset of $\mathbb{B}^q$ in which q is uniformly Lipschitz continuous in s with condition $\mathscr{L}_q$, i.e., $\mathbb{B}^q(\mathscr{L}_q) = \{q \in \mathbb{B}^q : |q(a', e, a, s) - q(a', e, a, s')| < \mathscr{L}_q |s - s'|, \forall (a', e, a), s, s'\}$. Similarly, $\mathbb{B}^\psi(\mathscr{L}_\psi) = \{\psi \in \mathbb{B}^\psi : |\psi^{(d,a')}(a', e, a, s) - \psi^{(d,a')}(a', e, a, s')| < \mathscr{L}_\psi |s - s'|, \forall (d, a'), (a', e, a), s, s'\}$. $\mathbb{B}^{\mathscr{W}}(\mathscr{L}_\psi) = \{\mathscr{W} \in \mathbb{B}^{\mathscr{W}} : |\mathscr{W}(i, e, a, s) - \mathscr{W}(i', e, a, s')| < \mathscr{L}_{\mathscr{W}} |s - s'|, \forall (i, e, a), s, s'\}$.

Define mapping $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3) : \mathbb{B}^q \times \mathbb{B}^\psi \times \mathbb{B}^{\mathscr{W}} \to \mathbb{B}^q \times \mathbb{B}^\psi \times \mathbb{B}^{\mathscr{W}}$ as following:

$$\begin{aligned} \mathbb{T}_1 q(a', e, a, s; q, \psi, \mathscr{W}) = \frac{1}{1+r} \sum_{e' \in \mathbb{E}} \{ & s[1 - m^{(1,0)}(g, e', a', \psi^{(0,a')}(e, a, s); q, \psi, \mathscr{W})] \\ & + (1 - s)[1 - m^{(1,0)}(b, e', a', \psi^{(0,a')}(e, a, s); q, \psi, \mathscr{W})]\}, \end{aligned} \tag{29}$$

$$\mathbb{T}_2 \psi^{(d,a')}(e, a, s; q, \psi, \mathscr{W}) = \xi^{(d,a')}(x; q, \psi, \mathscr{W}) \Omega(g|g) + (1 - \xi^{(d,a')}(x; q, \psi, \mathscr{W})) \Omega(g|b), \tag{30}$$

$$\mathbb{T}_3 \mathscr{W}(i, e, a, s; q, \psi, \mathscr{W}) = \gamma^C + \log[\sum_{(d,a') \in \mathbb{M}(x;q)} \exp(\frac{U^{(d,a')}(i, x; q)}{\sigma_\varepsilon} + \beta_i E \mathscr{W}(i', e', a', \psi^{(d,a')}(e, a, s))], \tag{31}$$

where,

$$m^{(d,a')}(i, e, a, s; q, \psi, \mathscr{W}) = \begin{cases} \dfrac{\exp(\frac{U^{(d,a')}(i,e,a,s;q)}{\sigma_\varepsilon} + \beta_i E \mathscr{W}(i', e', a', \psi^{(d,a')}(e, a, s))}{\sum_{(\tilde{d},\tilde{a'}) \in \mathbb{M}(e,a,s;q)} \exp(\frac{U^{(\tilde{d},\tilde{a'})}(i,e,a,s;q)}{\sigma_\varepsilon} + \beta_i E \mathscr{W}(i', e', \tilde{a'}, \psi^{(\tilde{d},\tilde{a'})}(e, a, s))))}, \text{ for } (d, a') \in \mathbb{M}(e, a, s; q), \\ 0, \text{ for } (d, a') \notin \mathbb{M}(e, a, s; q); \end{cases} \tag{32}$$

$$odd^{(d,a')}(e, a, s; q, \psi, \mathscr{W}) = \begin{cases} \dfrac{\exp(\beta_b E \mathscr{W}(b, e', a', \psi^{(d,a')}(e, a, s)))}{\exp(\beta_g E \mathscr{W}(g, e', a', \psi^{(d,a')}(e, a, s)))} \dfrac{\sum_{(\tilde{d},\tilde{a'}) \in \mathbb{M}(e,a,s;q)} \exp(\frac{U^{(\tilde{d},\tilde{a'})}(i,e,a,s;q)}{\sigma_\varepsilon} + \beta_g E \mathscr{W}(g, e', \tilde{a'}, \psi^{(\tilde{d},\tilde{a'})}(e, a, s))}{\sum_{(\tilde{d},\tilde{a'}) \in \mathbb{M}(e,a,s;q)} \exp(\frac{U^{(\tilde{d},\tilde{a'})}(i,e,a,s;q)}{\sigma_\varepsilon} + \beta_b E \mathscr{W}(b, e', \tilde{a'}, \psi^{(\tilde{d},\tilde{a'})}(e, a, s)))}, \text{ for } (d, a') = (0, a') \\ \dfrac{\exp(\frac{U^{(1,0)}(b,e,a,s)}{\sigma_\varepsilon} + \beta_b E \mathscr{W}(b, e', a', \psi^{(d,a')}(e, a, s)))}{\exp(\frac{U^{(1,0)}(g,e,a,s)}{\sigma_\varepsilon} + \beta_g E \mathscr{W}(g, e', a', \psi^{(d,a')}(e, a, s)))} \dfrac{\sum_{(\tilde{d},\tilde{a'}) \in \mathbb{M}(e,a,s;q)} \exp(\frac{U^{(\tilde{d},\tilde{a'})}(e,a,s,x;q)}{\sigma_\varepsilon} + \beta_g E \mathscr{W}(g, e', \tilde{a'}, \psi^{(\tilde{d},\tilde{a'})}(e, a, s))}{\sum_{(\tilde{d},\tilde{a'}) \in \mathbb{M}(e,a,s;q)} \exp(\frac{U^{(\tilde{d},\tilde{a'})}(e,a,s,x;q)}{\sigma_\varepsilon} + \beta_b E \mathscr{W}(b, e', \tilde{a'}, \psi^{(\tilde{d},\tilde{a'})}(e, a, s)))}, \text{ for } (d, a') = (1, 0) \end{cases} \tag{33}$$

$$\xi^{(d,a')}(e,a,s;q,\psi,\mathscr{W}) = \frac{1}{1+\frac{1-s}{s}\cdot odd^{(d,a')}(e,a,s;q,\psi,\mathscr{W})} \tag{34}$$

The goal is to find σ_ε large enough, s.t. $\exists(\mathscr{L}_q,\mathscr{L}_\psi,\mathscr{L}_\mathscr{W})$, the image of $\mathbb{T}$ is contained in $\mathbb{B}^q(\mathscr{L}_q)\times\mathbb{B}^\psi(\mathscr{L}_\psi)\times\mathbb{B}^\mathscr{W}(\mathscr{L}_\mathscr{W})$.

Lemma 6. *$\forall\sigma>1,\sigma_\varepsilon>0$, the function $h(c)=\exp(\frac{1}{\sigma_\varepsilon}\frac{c^{1-\sigma}}{1-\sigma})$ over $c\in(0,\bar{c}]$ satisfies following properties: (i) $\lim_{c\to 0+}h(c)=0$; (ii) $\lim_{c\to 0+}h'(c)=0$; (iii) $|h'(c)|\le\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}},\forall c\in(0,\bar{c}]$; (iv) $h(c)\le\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}}\bar{c}$.*

Proof. (i) is obvious. For (ii), consider

$$h'(c)=\frac{1}{\sigma_\varepsilon}c^{-\sigma}\exp(\frac{1}{\sigma_\varepsilon}\frac{c^{1-\sigma}}{1-\sigma}).$$

Consider the transformation $t=-\frac{1}{\sigma_\varepsilon}\frac{c^{1-\sigma}}{1-\sigma}$, then

$$\lim_{c\to 0+}h'(c)=\lim_{t\to+\infty}t^{\frac{\sigma}{\sigma-1}}\exp(-t)\cdot((\sigma-1)\sigma_\varepsilon)^{\frac{\sigma}{\sigma-1}},$$

which by L'Hospital's rule is equal to 0.

For (iii), consider

$$\begin{aligned}h''(c)&=\frac{-\sigma}{\sigma_\varepsilon}c^{-\sigma-1}\exp(\frac{1}{\sigma_\varepsilon}\frac{c^{1-\sigma}}{1-\sigma})+\frac{1}{\sigma_\varepsilon^2}c^{-2\sigma}\exp(\frac{1}{\sigma_\varepsilon}\frac{c^{1-\sigma}}{1-\sigma})\\&=\frac{1}{\sigma_\varepsilon}c^{-\sigma}\exp(\frac{1}{\sigma_\varepsilon}\frac{c^{1-\sigma}}{1-\sigma})(-\sigma c^{-1}+\frac{1}{\sigma_\varepsilon}c^{-\sigma}).\end{aligned}$$

Therefore, the maximum of $h'(c)$ is attained at $c^*=(\sigma\sigma_\varepsilon)^{\frac{1}{1-\sigma}}$, and

$$h'(c^*)=\frac{1}{\sigma_\varepsilon}(\sigma\sigma_\varepsilon)^{\frac{-\sigma}{1-\sigma}}\exp(\frac{\sigma}{1-\sigma})=\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}}.$$

By Mean Value Theorem, we get (iv). □

Now I show that $\mathbb{B}^\mathscr{W}$ can be chosen a set of functions with bounded value.

Lemma 7. *$\exists\underline{\mathscr{W}}(\sigma_\varepsilon)$ and $\overline{\mathscr{W}}(\sigma_\varepsilon)$ s.t. if $\underline{\mathscr{W}}(\sigma_\varepsilon)\le\mathscr{W}(i,e,a,s)\le\overline{\mathscr{W}}(\sigma_\varepsilon),\forall i,e,a,s$, then $\underline{\mathscr{W}}(\sigma_\varepsilon)\le\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W})\le\overline{\mathscr{W}}(\sigma_\varepsilon),\forall i,e,a,s;q,\psi$. Moreover, $\underline{\mathscr{W}}(\sigma_\varepsilon)$ can be chosen that is increasing in σ_ε; $\overline{\mathscr{W}}(\sigma_\varepsilon)$ can be chosen to be $\overline{\mathscr{W}}$ that does not depend on σ_ε.*

Proof. Since default is always an option,

$$\sum_{(d,a')\in\mathbb{M}(e,a,s)} \exp(\frac{U^{(d,a')}(e,a,s,x;q)}{\sigma_\varepsilon} + \beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s)))$$
$$\geq \exp(\frac{1}{\sigma_\varepsilon}\frac{[\phi_i(c^{(1,0)}(e,a,s)-v(e))]^{1-\sigma}}{1-\sigma})\cdot\exp(\beta_i\underline{\mathscr{W}}(\sigma_\varepsilon)).$$

Therefore,

$$\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W}) \geq \gamma^C + \frac{1}{\sigma_\varepsilon}\frac{[\phi_i(c^{(1,0)}(e,a,s)-v(e))]^{1-\sigma}}{1-\sigma} + \beta_i\underline{\mathscr{W}}(\sigma_\varepsilon).$$

Note I have used the property the consumption upon default $c^{(1,0)}(e,a,s)$ does not depend q. Thus fix i it suffices to set the lower bound s.t.

$$\underline{\mathscr{W}}(\sigma_\varepsilon) = \gamma^C + \frac{1}{\sigma_\varepsilon}\frac{[\phi_i(c^{(1,0)}(e,a,s)-v(e))]^{1-\sigma}}{1-\sigma} + \beta_i\underline{\mathscr{W}}(\sigma_\varepsilon).$$

Taking the lower across i we thus have

$$\underline{\mathscr{W}}(\sigma_\varepsilon) = \min_{i\in\{b,g\}}\frac{1}{1-\beta_i}(\gamma^C + \frac{1}{\sigma_\varepsilon}\frac{[\phi_i(c^{(1,0)}(e,a,s)-v(e))]^{1-\sigma}}{1-\sigma}).$$

Since $\sigma > 1$, we have $\underline{\mathscr{W}}(\sigma_\varepsilon)$ is increasing in σ_ε.

Next, since $\sigma > 1$, $U^{(d,a')}(i,e,a,s;q) < 0, \forall(i,e,a,s)$. Therefore,

$$\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W}) \leq \gamma^C + \log(N_{\mathbb{A}}+1) + \beta_i\overline{\mathscr{W}}(\sigma_\varepsilon),$$

where $N_{\mathbb{A}}$ is the cardinality of set $\mathbb{A}$. Therefore, it suffices to set

$$\overline{\mathscr{W}}(\sigma_\varepsilon) = \max_{i\in b,g}\frac{1}{1-\beta_i}(\gamma^C + \log(N_{\mathbb{A}}+1)),$$

which does not depend on σ_ε. $\square$

Next I establish Lipschitz condition for $\mathbb{T}_3\mathscr{W}$. I first show the following intermediate function is Lipschitz continuous. From now on denote $\mathscr{L}_q$, $\mathscr{L}_\psi$, $\mathscr{L}_\mathscr{W}$ as Lipschitz conditions of $q,\psi,\mathscr{W}$, respectively. The underlying assumptions in each following lemma is that $q,\psi,\mathscr{W}$ are Lipschitz continuous (note it does not mean I have imposed the image of $\mathbb{T}$ to be Lipschitz).

Consider function $\mu^{(d,a')}(i,e,a,s;q)$ defined as

$$\mu^{(d,a')}(i,e,a,s;q) = \begin{cases} \exp(\frac{1}{\sigma_\varepsilon}\frac{\tilde{c}^{(d,a')}(e,a,s;q)^{(1-\sigma)}}{1-\sigma}), & \text{for } (d,a') \in M(e,a,s;q), \\ 0, & \text{for } (d,a') \notin M(e,a,s;q). \end{cases}$$

Lemma 8. *$|\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s';q)| \leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \mathscr{L}_q C_1^\mu |s'-s|, \forall (d,a'), i, e, a, s,$ where C_1^μ is a constant that does not depend on σ_ε. Further, $|\mu^{(d,a')}(i,e,a,s;q)| \leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} C_0^\mu$ for some constant C_0^μ that does not depend on ε_ε.*

Proof. Case 1, If $(d,a') \in \mathbb{M}(e,a,s;q)$ and $(d,a') \in \mathbb{M}(e,a,s';q)$, using Lemma 6 and Mean Value Theorem,

$$\begin{aligned} |\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s';q)| &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} |\tilde{c}^{(d,a')}(e,a,s;q) - \tilde{c}^{(d,a')}(e,a,s';q)| \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} \max_{a' \in \mathbb{A}} |a'|\mathscr{L}_q|s'-s|. \end{aligned}$$

Case 2, suppose $(d,a') \in \mathbb{M}(e,a,s;q)$ and $(d,a') \notin \mathbb{M}(e,a,s';q)$. Notice in this case, $a' \neq 0$. Now I claim

$$\tilde{c}^{(d,a')}(e,a,s;q) \leq |s-s'||a'|\mathscr{L}_q$$

Suppose not, then we have

$$|s-s'| < \frac{\tilde{c}^{(d,a')}(e,a,s;q)}{|a'|\mathscr{L}_q},$$

which implies

$$\begin{aligned} |\tilde{c}^{(d,a')}(e,a,s';q) - \tilde{c}^{(d,a')}(e,a,s;q)| &\leq \mathscr{L}_q|a'||s'-s| \\ &< \tilde{c}^{(d,a')}(e,a,s;q), \end{aligned}$$

which implies $\tilde{c}^{(d,a')}(e,a,s';q) > 0$ that contradicts $(d,a') \notin \mathbb{M}(e,a,s';q)$. Therefore,

$$\begin{aligned} |\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s';q)| &= \mu^{(d,a')}(i,e,a,s;q) \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} |\tilde{c}^{(d,a')}(e,a,s;q) - 0| \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} \max_{a' \in \mathbb{A}} |a'|\mathscr{L}_q|s'-s| \end{aligned}$$

Setting $C_1^{\mu} = \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} \max_{a' \in \mathbb{A}} |a'|$ we have proved the first part.

For the second part, apply (iv) of Lemma 6 we have

$$|\mu^{(d,a')}(i,e,a,s;q)| \leq \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\bar{c}$$

where $\bar{c}$ can be any constant greater than the maximum total consumption attainable, which can be set as

$$\bar{c} = we_{max}n(e_{max})(1-\tau) - \frac{a_{min}}{1+r} + a_{max} + Tr - v(n(e_{max})).$$

Setting $C_0^{\mu} = \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}}\bar{c}$, we have proved the second part. □

We review the following properties about Lipschitz continuity.

Lemma 9. *Suppose functions $f(x), g(x)$ are defined on closed interval $[\underline{x}, \overline{x}]$ and are bounded in sup norm $|f|$ and $|g|$, and $f(x), g(x)$ are Lipschitz continuous in x with condition $\mathscr{L}_f$ and $\mathscr{L}_g$, respectively. Then the following are true:*

(1) $\mathscr{L}_{\alpha f \leq} |\alpha| \mathscr{L}_f$, $\forall \alpha \in R$;

(2) $\mathscr{L}_{f+g} \leq \mathscr{L}_f + \mathscr{L}_g$;

(3) $\mathscr{L}_{fg} \leq \mathscr{L}_f |g| + \mathscr{L} g |f|$;

(4) For function h that the compound evaluation $h(f)$ is defined, suppose $h(\cdot)$ is Lipschitz continuous with condition $\mathscr{L}_h$, then $\mathscr{L}_{h(f)} \leq \mathscr{L}_h \cdot \mathscr{L}_f$. If h is differentiable with bounded derivative $|h'|$, then $\mathscr{L}_{h(f)} \leq |h'| \mathscr{L}_f$.

Lemma 10. $|\mathbb{T}_3\mathscr{W}(i,e,a,s';q,\psi,\mathscr{W}) - \mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W})| < \mathscr{L}_{\mathbb{T}_3\mathscr{W}}|s'-s|$, *where*

$$\mathscr{L}_{\mathbb{T}_3\mathscr{W}} = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)\mathscr{L}_q + C_2^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi],$$

where $C_1^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon) > 0$ is decreasing in σ_ε, and $C_2^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon) > 0$ is decreasing in σ_ε.

Proof. First, default is always an option so the sum of exponential value is bounded below, denoted by $LB(\sigma_\varepsilon)$:

$$\sum_{(d,a')\in\mathbb{M}(e,a,s;q)} \exp(\frac{U^{(d,a')}(i,e,a,s;q)}{\sigma_\varepsilon} + \beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s)))$$
$$\geq \min_{i\in\{b,g\}} \exp(\frac{1}{\sigma_\varepsilon}\frac{[Tr+(1-\tau)we_{min}n(e_{min})-n(e_{min})]^{1-\sigma}}{1-\sigma} + \beta_i\underline{\mathscr{W}}(\sigma_\varepsilon))$$
$$= LB(\sigma_\varepsilon).$$

Notice that $LB(\sigma_\varepsilon)$ is increasing in σ_ε since $\sigma > 1$ and $\underline{\mathscr{W}}(\sigma_\varepsilon)$ is increasing in σ_ε (Lemma 7).

Denote $\mathscr{L}(f)$ as the Lipschitz condition for the function f w.r.t. s. Notice

$$\sum_{(d,a')\in\mathbb{M}(x;q)} \exp(\frac{U^{(d,a')}(i,e,a,s;q)}{\sigma_\varepsilon} + \beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s))$$

can be written as

$$\sum_{(d,a')\in\mathbb{Y}} \mu^{(d,a')}(i,e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s))),$$

because $\mu^{(d,a')}(i,e,a,s;q)$ assigns 0 to $(d,a')\notin\mathbb{M}(e,a,s;q)$.

Using Lemma 9,

$$\mathscr{L}(\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s)))) \leq \beta_i \exp(\beta_i\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}.$$

And using Lemma 8,

$$\mathscr{L}(\mu^{(d,a')}(i,e,a,s;q)) \leq C_1^\mu \sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q,$$

$$|\mu^{(d,a')}(i,e,a,s;q)| \leq C_0^\mu \sigma_\varepsilon^{\frac{1}{1-\sigma}},$$

Therefore, using Lemma 9,

$$\begin{aligned}
&\mathscr{L}(\mu^{(d,a')}(i,e,a,s;q)\cdot\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(i,e,a,s))))\\
C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}&\mathscr{L}(\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s))))+\exp(\beta_i\mathscr{W})\mathscr{L}(\mu^{(d,a')}(i,e,a,s;q))\\
&\leq \beta_i C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_i\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}+C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q.\\
&\leq \beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}+C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{L}[\sum_{(d,a')\in\mathbb{M}(x;q)}&\exp(\frac{U^{(d,a')}(i,e,a,s;q)}{\sigma_\varepsilon}+\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(i,e,a,s)))]\\
&\leq [\beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}+C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q](N_{\mathbb{A}}+1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathscr{L}[\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W})|\\
=\mathscr{L}[\log(\sum_{(d,a')\in\mathbb{M}(x;q)}&\exp(\frac{U^{(d,a')}(i,e,a,s;q)}{\sigma_\varepsilon}+\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,x))))]\\
&\leq\frac{1}{LB(\sigma_\varepsilon)}[\beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}+C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q](N_{\mathbb{A}}+1)
\end{aligned}$$

Setting

$$C_1^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)=\frac{1}{LB(\sigma_\varepsilon)}\beta_g C_0^{\mu}\exp(\beta_g\overline{\mathscr{W}})(N_{\mathbb{A}}+1)$$

$$C_2^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)=\frac{1}{LB(\sigma_\varepsilon)}C_1^{\mu}(N_{\mathbb{A}}+1).$$

Since $LB_1(\sigma_\varepsilon)>0$ and is increasing in σ_ε, we have proved the results. □

Lemma 11. *m defined in Equation 32 is Lipschitz continuous in s with condition*

$$\mathscr{L}_m=\sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^m(\sigma_\varepsilon)\mathscr{L}_q+C_2^m(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}],$$

where $C_1^m(\sigma_\varepsilon)>0$ is decreasing in σ_ε, and $C_2^m(\sigma_\varepsilon)>0$ is decreasing in σ_ε.

Proof. Notice

$$m^{(d,a')}(i,e,a,s;q,\psi,\mathscr{W}) = \frac{\mu^{(d,a')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s))}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{Y}}\mu^{(\tilde{d},\tilde{a}')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',\tilde{a}',\psi^{(\tilde{d},\tilde{a}')}(e,a,s)))}.$$

As shown in Lemma 10,

$$\mathscr{L}[\mu^{(d,a')}(i,e,a,s;q)\cdot\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s)))] \leq \beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi} + C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q.$$

$$\begin{aligned}&\mathscr{L}[\sum_{(\tilde{d},\tilde{a}')\in\mathbb{Y}}\mu^{(\tilde{d},\tilde{a}')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',\tilde{a}',\psi^{(\tilde{d},\tilde{a}')}(e,a,s)))]\\ &\quad\leq[\beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi} + C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q](N_{\mathbb{A}}+1)\end{aligned}$$

$$\sum_{(\tilde{d},\tilde{a}')\in\mathbb{Y}}\mu^{(\tilde{d},\tilde{a}')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',\tilde{a}',\psi^{(\tilde{d},\tilde{a}')}(e,a,s)))\geq LB(\sigma_\varepsilon)$$

Apply Lemma 9,

$$\begin{aligned}\mathscr{L}(m^{(d,a')}(i,e,a,s;q,\psi,\mathscr{W})) &\leq \frac{1}{LB^2(\sigma_\varepsilon)}[\beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi} + C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q](N_{\mathbb{A}}+1)\exp(\beta_g\overline{\mathscr{W}})\\ &\quad+[\beta_g C_0^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\exp(\beta_g\overline{\mathscr{W}})\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi} + C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q]\frac{1}{LB(\sigma_\varepsilon)}\\ &= [\frac{1}{LB^2(\sigma_\varepsilon)}(N_{\mathbb{A}}+1)\exp(\beta_g\overline{\mathscr{W}})C_0^{\mu} + \frac{1}{LB(\sigma_\varepsilon)}C_0^{\mu}]\beta_g\exp(\beta_g\overline{\mathscr{W}})\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_{\mathscr{W}}\mathscr{L}_{\psi}\\ &\quad+[\frac{1}{LB^2(\sigma_\varepsilon)}(N_{\mathbb{A}}+1)\exp(\beta_g\overline{\mathscr{W}}) + \frac{1}{LB(\sigma_\varepsilon)}]C_1^{\mu}\sigma_\varepsilon^{\frac{1}{1-\sigma}}\mathscr{L}_q\end{aligned}$$

Setting

$$C_1^m(\sigma_\varepsilon) = [\frac{1}{LB^2(\sigma_\varepsilon)}(N_{\mathbb{A}}+1)\exp(\beta_g\overline{\mathscr{W}}) + \frac{1}{LB(\sigma_\varepsilon)}]C_1^{\mu}$$

and

$$C_2^m(\sigma_\varepsilon) = [\frac{1}{LB^2(\sigma_\varepsilon)}(N_{\mathbb{A}}+1)\exp(\beta_g\overline{\mathscr{W}}) + \frac{1}{LB(\sigma_\varepsilon)}]C_0^{\mu}\beta_g\exp(\beta_g\overline{\mathscr{W}}).$$

Since $LB(\sigma_\varepsilon)$ is increasing in σ_ε, we have proved the result. □

Similarly we can show

Lemma 12. *odd defined in Equation 33 is Lipschitz continuous in s with condition*

$$\mathscr{L}_{odd} = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{odd}(\sigma_\varepsilon)\mathscr{L}_q + C_2^{odd}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi],$$

where $C_1^{odd}(\sigma_\varepsilon) > 0$ *is decreasing in* σ_ε*, and* $C_2^{odd}(\sigma_\varepsilon) > 0$ *is decreasing in* σ_ε*.*

Lemma 13. ξ *defined in Equation 34 is Lipschitz continuous in s with condition*

$$\mathscr{L}_\xi = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^\xi(\sigma_\varepsilon)\mathscr{L}_q + C_2^\xi(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi] + C_3^\xi,$$

where $C_1^\xi(\sigma_\varepsilon) > 0$ *is decreasing in* σ_ε*,* $C_2^\xi(\sigma_\varepsilon) > 0$ *is decreasing in* σ_ε*, and* $C_3^\xi > 0$ *does not depend on* σ_ε*.*

Proof. Notice $0 < \underline{s} \leq s \leq \bar{s} < 1$,

$$\frac{1}{\bar{s}} - 1 \leq \frac{1-s}{s} \leq \frac{1}{\underline{s}} - 1$$

and

$$|\frac{d\frac{1-s}{s}}{ds}| = |\frac{1}{s^2}| \in [\frac{1}{\bar{s}^2}, \frac{1}{\underline{s}^2}]$$

Using Lemma 9,

$$\begin{aligned} \mathscr{L}[\frac{1-s}{s} \cdot odd^{(d,a')}(e,a,s;q,\psi,\mathscr{W})] \\ \leq \frac{1}{\underline{s}^2} + \mathscr{L}_{odd}(\frac{1}{\underline{s}} - 1) \end{aligned}$$

$$\begin{aligned} \mathscr{L}[\frac{1}{1+\frac{1-s}{s}\cdot odd^{(d,a')}(e,a,s;q,\psi,\mathscr{W})}] &\leq \frac{1}{\underline{s}^2} + \mathscr{L}_{odd}(\frac{1}{\underline{s}} - 1) \\ &\leq \frac{1}{\underline{s}^2} + (\frac{1}{\underline{s}} - 1)(\sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{odd}(\sigma_\varepsilon)\mathscr{L}_q + C_2^{odd}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi]). \end{aligned}$$

Then we set

$$C_1^\xi(\sigma_\varepsilon) = (\frac{1}{\underline{s}} - 1)C_1^{odd}(\sigma_\varepsilon)$$

$$C_2^\xi(\sigma_\varepsilon) = (\frac{1}{\underline{s}} - 1)C_2^{odd}(\sigma_\varepsilon)$$

$$C_3^{\xi} = \frac{1}{\underline{s}^2}$$

□

Lemma 14. *$\mathbb{T}_2\psi$ defined in Equation 30 is Lipschitz continuous in s with condition*

$$\mathscr{L}_{\mathbb{T}_2\psi} = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{\mathbb{T}_2\psi}(\sigma_\varepsilon)\mathscr{L}_q + C_2^{\mathbb{T}_2\psi}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi] + C_3^{\mathbb{T}_2\psi},$$

where $C_1^{\mathbb{T}_2\psi}(\sigma_\varepsilon) > 0$ is decreasing in σ_ε, $C_2^{\mathbb{T}_2\psi}(\sigma_\varepsilon) > 0$ is decreasing in σ_ε, and $C_3^{\mathbb{T}_2\psi} > 0$ does not depend on σ_ε.

Proof. Apply Lemma 9,

$$\begin{aligned}\mathscr{L}_{\mathbb{T}_2\psi} &\leq (\Omega(g|g) + \Omega(g|b))\mathscr{L}_\xi \\ &\leq (\Omega(g|g) + \Omega(g|b))[\sigma_\varepsilon^{\frac{1}{1-\sigma}}(C_1^{\xi}(\sigma_\varepsilon)\mathscr{L}_q + C_2^{\xi}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi) + C_3^{\xi}].\end{aligned}$$

□

Lemma 15. *$\mathbb{T}q$ defined in Equation 29 is Lipschitz continuous in s with condition*

$$\mathscr{L}_{\mathbb{T}_1 q} = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{\mathbb{T}_1 q}(\sigma_\varepsilon)\mathscr{L}_q\mathscr{L}_\psi + C_2^{\mathbb{T}_1 q}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi^2] + C_3^{\mathbb{T}_1 q},$$

where $C_1^{\mathbb{T}_1 q}(\sigma_\varepsilon) > 0$ is decreasing in σ_ε, $C_2^{\mathbb{T}_1 q}(\sigma_\varepsilon) > 0$ is decreasing in σ_ε, and $C_3^{\mathbb{T}_1 q} > 0$ does not depend on σ_ε.

Proof. Apply Lemma 9,

$$\begin{aligned}\mathscr{L}_{\mathbb{T}_1 q} &\leq \frac{1}{1+r}(\bar{s}\mathscr{L}_m\mathscr{L}_\psi + 1 + (1-\underline{s})\mathscr{L}_m\mathscr{L}_\psi + 1) \\ &\leq \frac{2}{1+r} + (1+\bar{s}-\underline{s})\mathscr{L}_\psi[\sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^m(\sigma_\varepsilon)\mathscr{L}_q + C_2^m(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi]].\end{aligned}$$

□

Lemma 16. *$\forall \mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_{\mathscr{W}}$, $\exists \sigma_3(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_{\mathscr{W}})$ s.t. $\forall \sigma_\varepsilon > \sigma_3(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_{\mathscr{W}})$, $\mathscr{L}_{\mathbb{T}_3\mathscr{W}} < \mathscr{L}_{\mathscr{W}}$.*

Proof. From Lemma 10,

$$\mathscr{L}_{\mathbb{T}_3\mathscr{W}} = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)\mathscr{L}_q + C_2^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)\mathscr{L}_{\mathscr{W}}\mathscr{L}_\psi].$$

Choose $\sigma_3(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$ s.t.

$$\sigma_3^{\frac{1}{1-\sigma}} C_2^{\mathbb{T}_3\mathscr{W}}(\sigma_1)\mathscr{L}_\psi < 1/2$$

and

$$\sigma_3^{\frac{1}{1-\sigma}} C_1^{\mathbb{T}_3\mathscr{W}}(\sigma_1)\mathscr{L}_q < \frac{1}{2}\mathscr{L}_\mathscr{W}$$

This can be done since $\lim_{\sigma_\varepsilon\to\infty}\sigma_\varepsilon^{\frac{1}{1-\sigma}} = 0$ and $C_1^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)$ and $C_2^{\mathbb{T}_3\mathscr{W}}(\sigma_\varepsilon)$ are decreasing in σ_ε.

Then we have $\forall\sigma_\varepsilon > \sigma_3(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$, $\mathscr{L}_{\mathbb{T}_3\mathscr{W}} < \mathscr{L}_\mathscr{W}$. □

Lemma 17. *$\forall\mathscr{L}_q, \mathscr{L}_\psi \leq 2C_3^{\mathbb{T}_2\psi}, \mathscr{L}_\mathscr{W}$, $\exists\sigma_2(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$ s.t. $\forall\sigma_\varepsilon > \sigma_2(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$, $\mathscr{L}_{\mathbb{T}_2\psi} < 2C_3^{\mathbb{T}_2\psi}$.*

Proof. From Lemma 14,

$$\mathscr{L}_{\mathbb{T}_1 q} = \sigma_\varepsilon^{\frac{1}{1-\sigma}}[C_1^{\mathbb{T}_1 q}(\sigma_\varepsilon)\mathscr{L}_q\mathscr{L}_\psi + C_2^{\mathbb{T}_1 q}(\sigma_\varepsilon)\mathscr{L}_\mathscr{W}\mathscr{L}_\psi^2] + C_3^{\mathbb{T}_1 q}.$$

$\forall\mathscr{L}_q, \mathscr{L}_\psi < 2C_3^{\mathbb{T}_2\psi}, \mathscr{L}_\mathscr{W}$, choose $\sigma_2(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$ s.t. $\sigma_2^{\frac{1}{1-\sigma}}[C_1^{\mathbb{T}_1 q}(\sigma_2)\mathscr{L}_q\mathscr{L}_\psi + C_2^{\mathbb{T}_1 q}(\sigma_2)\mathscr{L}_\mathscr{W}\mathscr{L}_\psi^2] < C_3^{\mathbb{T}_1 q}$. This can be done since $\lim_{\sigma_\varepsilon\to\infty}\sigma_\varepsilon^{\frac{1}{1-\sigma}} = 0$ and $C_1^{\mathbb{T}_2\psi}(\sigma_\varepsilon)$ and $C_2^{\mathbb{T}_2\psi}(\sigma_\varepsilon)$ are decreasing in σ_ε. □

Similarly, we can establish

Lemma 18. *$\forall\mathscr{L}_q \leq 2C_3^{\mathbb{T}_1 q}, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W}$, $\exists\sigma_1(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$ s.t. $\forall\sigma_\varepsilon > \sigma_1(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$, $\mathscr{L}_{\mathbb{T}_1 q} < 2C_3^{\mathbb{T}_1 q}$.*

Through the above three lemmas, we have the following

Lemma 19. *For $\mathscr{L}_q = 2C_3^{\mathbb{T}_1 q}$, $\mathscr{L}_\psi = 2C_3^{\mathbb{T}_2\psi}$, $\mathscr{L}_\mathscr{W} = 1$, $\exists\sigma_\varepsilon^*$ s.t. $\forall\sigma_\varepsilon > \sigma_\varepsilon^*$, $\mathscr{L}_{\mathbb{T}_1 q} \leq 2C_3^{\mathbb{T}_1 q}$, $\mathscr{L}_{\mathbb{T}_2\psi} \leq 2C_3^{\mathbb{T}_2\psi}$, $\mathscr{L}_{\mathbb{T}_3\mathscr{W}} \leq 1$.*

Proof. Choose $\sigma_1(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W}), \sigma_2(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W}), \sigma_3(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})$ from Lemma 18, Lemma 17, and Lemma 16, respectively.

Choose $\sigma_\varepsilon^* = \max\{\sigma_1(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W}), \sigma_2(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W}), \sigma_3(\mathscr{L}_q, \mathscr{L}_\psi, \mathscr{L}_\mathscr{W})\}$. Then the result follows Lemma 18, 17, and 16. □

Up to now we have established that by choosing σ_ε large enough, $\mathbb{T}$ maps $\mathbb{B}^q(\mathscr{L}_q) \times \mathbb{B}^\psi(\mathscr{L}_\psi) \times \mathbb{B}^\mathscr{W}(\mathscr{L}_\mathscr{W})$ to itself. Now we prove the mapping $\mathbb{T}$ is continuous. Denote $||\cdot||$ as sup norm.

Lemma 20. *The intermediate function $\mu^{(d,a')}(i,e,a,s;q)$ defined in Lemma 8 is continuous in q with sup norm for $q \in B^q(\mathscr{L}_q)$.*

Proof. Consider

$$|\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s;q')|$$

under different cases.

Case 1: $(d,a') \in \mathbb{M}(e,a,s;q)$, and $(d,a') \in \mathbb{M}(e,a,s;q')$, by Mean Value Theorem

$$\begin{aligned}|\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s;q')| &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} |\tilde{c}^{(d,a')}(e,a,s;q) - \tilde{c}^{(d,a')}(e,a,s;q')| \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} \max_{a' \in \mathbb{A}} |a'| \cdot ||q - q'||\end{aligned}$$

Case 2: $(d,a') \in \mathbb{M}(e,a,s;q)$, and $(d,a') \notin \mathbb{M}(e,a,s;q')$, Notice in this case, $a' \neq 0$. Now I claim

$$\tilde{c}^{(d,a')}(e,a,s;q) \leq |q(a',e,a,s) - q'(a',e,a,s)| \cdot |a'|$$

Suppose not, then we have

$$|q(a',e,a,s) - q'(a',e,a,s)| < \frac{\tilde{c}^{(d,a')}(e,a,s;q)}{|a'|},$$

which implies

$$\begin{aligned}|\tilde{c}^{(d,a')}(e,a,s;q') - \tilde{c}^{(d,a')}(e,a,s;q)| &\leq |q(a',e,a,s) - q'(a',e,a,s)| \cdot |a'| \\ &< \tilde{c}^{(d,a')}(e,a,s;q),\end{aligned}$$

which implies $\tilde{c}^{(d,a')}(e,a,s;q') > 0$ that contradicts $(d,a') \notin \mathbb{M}(e,a,s;q')$. Therefore,

$$\begin{aligned}|\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s;q')| &= |\mu^{(d,a')}(i,e,a,s;q) - 0| \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} |\tilde{c}^{(d,a')}(e,a,s;q) - 0| \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} |q(a',e,a,s) - q'(a',e,a,s)| \cdot |a'| \\ &\leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} \max_{a' \in \mathbb{A}} |a'| \cdot ||q - q'||.\end{aligned}$$

Similarly for case 3 where $(d,a') \notin \mathbb{M}(e,a,s;q)$, and $(d,a') \in \mathbb{M}(e,a,s;q')$. And for case 4 $(d,a') \notin \mathbb{M}(e,a,s;q)$ and $(d,a') \notin \mathbb{M}(e,a,s;q')$ we have

$$|\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s;q')| = 0.$$

Summarize over all four cases we have

$$||\mu^{(d,a')}(i,e,a,s;q) - \mu^{(d,a')}(i,e,a,s;q')|| \leq \sigma_\varepsilon^{\frac{1}{1-\sigma}} \exp(\frac{\sigma}{1-\sigma})\sigma^{\frac{-\sigma}{1-\sigma}} \max_{a'\in\mathbb{A}} |a'| \cdot ||q-q'||.$$

Therefore $\mu^{(d,a')}(i,e,a,s;q)$ is continuous in q with sup norm. $\square$

Lemma 21. *$\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W})$ is continuous in $(q,\psi,\mathscr{W})$ with sup norm on $\mathbb{B}^q(\mathscr{L}_q) \times \mathbb{B}^\psi(\mathscr{L}_\psi) \times \mathbb{B}^\mathscr{W}(\mathscr{L}_\mathscr{W})$.*

Proof. Write $\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W})$ as

$$\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W}) = \gamma^C + \log(\sum_{(d,a')\in\mathbb{Y}} \mu^{(d,a')}(i,e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s))))$$

Since $\sum_{(d,a')\in\mathbb{Y}} \mu^{(d,a')}(i,e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s)))$ is uniformly bounded positive below (by $LB(\sigma_\varepsilon)$ established in Lemma 10), $\mu^{(d,a')}(i,e,a,s;q)$ is continuous in q, $\mathscr{W}(i,e,a,s)$ is Lipschitz continuous in s,
therefore, $\mathbb{T}_3\mathscr{W}(i,e,a,s;q,\psi,\mathscr{W})$ is continuous in $(q,\psi,\mathscr{W})$. $\square$

Lemma 22. *$m^{(d,a')}(i,e,a,s;q,\psi,\mathscr{W})$ defined in Equation 32 is continuous in $(q,\psi,\mathscr{W})$ with sup norm on $\mathbb{B}^q(\mathscr{L}_q) \times \mathbb{B}^\psi(\mathscr{L}_\psi) \times \mathbb{B}^\mathscr{W}(\mathscr{L}_\mathscr{W})$.*

Proof. Write $m^{(d,a')}(i,e,a,s;q,\psi,\mathscr{W})$ as following

$$m^{(d,a')}(i,e,a,s;q,\psi,\mathscr{W}) = \frac{\mu^{(d,a')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',a',\psi^{(d,a')}(e,a,s))}{\sum_{(\tilde{d},\tilde{a}')\in\mathbb{Y}} \mu^{(\tilde{d},\tilde{a}')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',\tilde{a}',\psi^{(\tilde{d},\tilde{a}')}(e,a,s)))}.$$

Note the following properties. (1) $\mathscr{W}$ is uniformly bounded above and below. (2) $\mathscr{W}$ is Lipschitz in s. (3) μ is continuous in q.
(4) $\sum_{(\tilde{d},\tilde{a}')\in\mathbb{Y}} \mu^{(\tilde{d},\tilde{a}')}(e,a,s;q)\exp(\beta_i E\mathscr{W}(i',e',\tilde{a}',\psi^{(\tilde{d},\tilde{a}')}(e,a,s)))$ is uniformly bounded above and uniformly positively bounded below (established in the proof of Lemma 7). Therefore, we have m is continuous in $(q,\psi,\mathscr{W})$. $\square$

We next establish

Lemma 23. *$\mathbb{T}_1 q(a',e,a,s;q,\psi,\mathscr{W})$ is continuous in $(q,\psi,\mathscr{W})$ with sup norm on $\mathbb{B}^q(\mathscr{L}_q) \times \mathbb{B}^\psi(\mathscr{L}_\psi) \times \mathbb{B}^\mathscr{W}(\mathscr{L}_\mathscr{W})$.*

Proof. Consider $(q_n, \psi_n, \mathscr{W}_n) \to (q, \psi, \mathscr{W})$ with sup norm, we have

$$\begin{aligned}
&\max_{i,e',a',e,a,s} |m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q,\psi,\mathscr{W}) - m^{(d,a')}(i,e',a',\psi_n^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n)| \\
&\leq \max_{i,e',a',e,a,s} |m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q,\psi,\mathscr{W}) - m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n)| \\
&+ \max_{i,e',a',e,a,s} |m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n) - m^{(d,a')}(i,e',a',\psi_n^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n)|
\end{aligned}$$

First,

$$\lim_{n\to\infty} \max_{i,e',a',e,a,s} |m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q,\psi,\mathscr{W}) - m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n)| = 0$$

since $m(\cdot;q,\psi,\mathscr{W})$ is continuous in $(q,\psi,\mathscr{W})$. Second,

$$\begin{aligned}
&|m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n) - m^{(d,a')}(i,e',a',\psi_n^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n)| \\
&\qquad \leq \mathscr{L}_m ||\psi - \psi_n||
\end{aligned}$$

since $m^{(d,a')}(i,e,a,s;q,\psi,\mathscr{W})$ is uniformly Lipschitz continuous in s with condition $\mathscr{L}_m$, as established in Lemma 11.

Therefore,

$$\lim_{n\to\infty} \max_{i,e',a',e,a,s} |m^{(d,a')}(i,e',a',\psi^{(0,a')}(e,a,s);q,\psi,\mathscr{W}) - m^{(d,a')}(i,e',a',\psi_n^{(0,a')}(e,a,s);q_n,\psi_n,\mathscr{W}_n)| = 0$$

Therefore, consider $\mathbb{T}_1 q(\cdot;q,\psi,\mathscr{W})$

$$\begin{aligned}
\mathbb{T}_1 q(a',e,a,s;q,\psi,\mathscr{W}) = \frac{1}{1+r} \sum_{e'\in\mathbb{E}} \{ & s[1 - m^{(1,0)}(g,e',a',\psi^{(0,a')}(e,a,s);q,\psi,\mathscr{W})] \\
& +(1-s)[1 - m^{(1,0)}(b,e',a',\psi^{(0,a')}(e,a,s);q,\psi,\mathscr{W})]\},
\end{aligned}$$

$$\lim_{n\to\infty} \max_{a',e,a,s} |\mathbb{T}_1 q(a',e,a,s;q,\psi,\mathscr{W}) - \mathbb{T}_1 q(a',e,a,s;q_n,\psi_n,\mathscr{W}_n)| = 0$$

□

Similarly we can prove

Lemma 24. *$\mathbb{T}_2\psi(a',e,a,s;q,\psi,\mathscr{W})$ is continuous in $(q,\psi,\mathscr{W})$ with sup norm on $\mathbb{B}^q(\mathscr{L}_q) \times \mathbb{B}^\psi(\mathscr{L}_\psi) \times \mathbb{B}^\mathscr{W}(\mathscr{L}_\mathscr{W})$.*

We next apply Schauder fixed point theorem on the mapping $\mathbb{T}$.

Lemma 25. *$\exists \sigma_\varepsilon^*$ s.t. $\forall \sigma_\varepsilon > \sigma_\varepsilon^*$, $\mathbb{T}$ defined in Equation 29, 30, and 31 has fixed points*[22].

Proof. First, from Lemma 7, for σ_ε large enough, $\mathbb{B}^{\mathscr{W}}$ can be chosen the set of functions uniformly bounded by $\underline{\mathscr{W}}(1)$ and $\overline{\mathscr{W}}$. (Notice $\underline{\mathscr{W}}(\sigma_\varepsilon)$ is increasing in σ_ε).

Fix $\mathscr{L}_q = 2C_3^{\mathbb{T}_1 q}$, $\mathscr{L}_\psi = 2C_3^{\mathbb{T}_2\psi}$, $\mathscr{L}_{\mathscr{W}} = 1$. From Lemma 19, $\exists \sigma_\varepsilon^*$ s.t. $\forall \sigma_\varepsilon > \sigma_\varepsilon^*$, $\mathscr{L}_{\mathbb{T}_1 q} \leq 2C_3^{\mathbb{T}_1 q}$, $\mathscr{L}_{\mathbb{T}_2\psi} \leq 2C_3^{\mathbb{T}_2\psi}$, $\mathscr{L}_{\mathbb{T}_3\mathscr{W}} \leq 1$.

Now denote $\tilde{\mathbb{T}} : \mathbb{B}^q(2C_3^{\mathbb{T}_1 q}) \times \mathbb{B}^\psi(2C_3^{\mathbb{T}_2\psi}) \times \mathbb{B}^{\mathscr{W}}(1) \to \mathbb{B}^q \times \mathbb{B}^\psi \times \mathbb{B}^{\mathscr{W}}$ as mapping that agrees with $\mathbb{T}$ on the subset $\mathbb{B}^q(2C_3^{\mathbb{T}_1 q}) \times \mathbb{B}^\psi(2C_3^{\mathbb{T}_2\psi}) \times \mathbb{B}^{\mathscr{W}}(1)$.

Then $\tilde{\mathbb{T}}(\mathbb{B}^q(2C_3^{\mathbb{T}_1 q}) \times \mathbb{B}^\psi(2C_3^{\mathbb{T}_2\psi}) \times \mathbb{B}^{\mathscr{W}}(1)) \subseteq \mathbb{B}^q(2C_3^{\mathbb{T}_1 q}) \times \mathbb{B}^\psi(2C_3^{\mathbb{T}_2\psi}) \times \mathbb{B}^{\mathscr{W}}(1)$. And from Lemma 21, 23, and 24, $\tilde{\mathbb{T}}$ is continuous.

By Arzelà–Ascoli theorem, the set of bounded Lipschitz functions $\mathbb{B}^q(2C_3^{\mathbb{T}_1 q}) \times \mathbb{B}^\psi(2C_3^{\mathbb{T}_2\psi}) \times \mathbb{B}^{\mathscr{W}}(1)$ is compact. All conditions applying Schauder fixed point theorem are satisfied. Therefore, $\tilde{\mathbb{T}}$ has fixed points, which are also fixed points of $\mathbb{T}$. □

Now given $q, \psi, \mathscr{W}$ and the induced policy function $m(\cdot; q, \psi, \mathscr{W})$, I establish the existence of stationary probability measure.

Denote M^Φ as the set of probability measure Φ over $(\mathbb{I} \times \mathbb{E} \times \mathbb{A} \times \mathbb{S}, \mathcal{P}(\mathbb{I}) \times \mathcal{P}(\mathbb{E}) \times \mathcal{P}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}))$, where $\mathcal{P}(\cdot)$ is the power set of underlying discrete sets, and $\mathcal{B}(\mathbb{S})$ is the Borel algebra of $[\underline{s}, \bar{s}]$. Denote $\mathbb{T}_4 : M^\Phi \to M^\Phi$ defined as following:

$$\begin{aligned}\mathbb{T}_4\Phi(i', e', a', S'; \Phi) = &\int \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) \sum_{a'} m^{(0,a')}(i, e, a, s; q, \psi, \mathscr{W}) \mathbb{1}(\psi^{(0,a')}(e, a, s) \in S')\Phi(i, e, a, ds) \\ &+ \mathbb{1}(a' = 0) \int \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) m^{(1,0)}(i, e, a, s; q, \psi, \mathscr{W}) \mathbb{1}(\psi^{(1,0)}(e, a, s) \in S')\Phi(i, e, a, ds), \\ &\forall i', e', a', S' \in \mathcal{B}(\mathbb{S})\end{aligned} \tag{35}$$

We prove $\mathbb{T}_4$ is continuous in Φ under the weak topology. Similar proof strategies have been used in Cao (2016).

We first have the following:

Lemma 26. *$\forall f$ Lipschitz continuous (with condition $\mathscr{L}_f$) and sequence of Φ_n that is weakly converging to Φ*

$$\lim_{n\to\infty} \int_{s' \in S'} f(s') T_4\Phi(i', e', a', ds'; \Phi_n) = \int_{s' \in S'} f(s') T_4\Phi(i', e', a', ds'; \Phi)$$

[22] σ_ε^* depends on (r, τ, Tr) and other model parameters.

$\forall i', e', a', S' \in \mathcal{B}(\mathbb{S})$.

Proof. Consider

$$\lim_{n\to\infty} \int_{s'\in S'} f(s') T_4 \Phi(i', e', a', ds'; \Phi_n)$$

$$= \lim_{n\to\infty} \int_{s'\in S'} f(s')[\int_s \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) \sum_{a'} m^{(0,a')}(i,e,a,s;q,\psi,\mathscr{W})\mathbb{1}(\psi^{(0,a')}(e,a,s) \in ds')\Phi_n(i,e,a,ds)$$

$$+\mathbb{1}(a'=0) \int_s \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) m^{(1,0)}(i,e,a,s;q,\psi,\mathscr{W})\mathbb{1}(\psi^{(1,0)}(e,a,s) \in ds')\Phi_n(i,e,a,ds)]$$

$$= \lim_{n\to\infty} [\int_s \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) \sum_{a'} m^{(0,a')}(i,e,a,s;q,\psi,\mathscr{W}) \int_{s'\in S'} f(s')\mathbb{1}(\psi^{(0,a')}(e,a,s) \in ds')\Phi_n(i,e,a,ds)$$

$$+\mathbb{1}(a'=0) \int_s \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) m^{(1,0)}(i,e,a,s;q,\psi,\mathscr{W}) \int_{s'\in S'} f(s')\mathbb{1}(\psi^{(1,0)}(e,a,s) \in ds')\Phi_n(i,e,a,ds)]$$

$$= \lim_{n\to\infty} [\int_{\{s:\psi^{(0,a')}(e,a,s)\in S'\}} \sum_{i,e,a} \Omega(i'|i)\Gamma(e'|e) \sum_{a'} m^{(0,a')}(i,e,a,s;q,\psi,\mathscr{W}) f(\psi^{(0,a')}(e,a,s))\Phi_n(i,e,a,ds)$$

$$+\mathbb{1}(a'=0) \sum_{i,e,a} \int_{\{s:\psi^{(0,a')}(e,a,s)\}\in S'} \Omega(i'|i)\Gamma(e'|e) m^{(1,0)}(i,e,a,s;q,\psi,\mathscr{W}) f(\psi^{(0,a')}(e,a,s))\Phi_n(i,e,a,ds)]$$

The second equal changes order of integration since the inner integral is finite (m is positive bounded below 1).

Since f, m, ψ are all bounded and Lipschitz in s (f is bounded because f is Lipschitz and defined on a compact set), by definition of weak convergence the above

$$= \sum_{i,e,a} \int_{\{s:\psi^{(0,a')}(e,a,s)\in S'\}} \Omega(i'|i)\Gamma(e'|e) \sum_{a'} m^{(0,a')}(i,e,a,s;q,\psi,\mathscr{W}) f(\psi^{(0,a')}(e,a,s))\Phi(i,e,a,ds)$$

$$+\mathbb{1}(a'=0) \sum_{i,e,a} \int_{\{s:\psi^{(0,a')}(e,a,s)\in S'\}} \Omega(i'|i)\Gamma(e'|e) m^{(1,0)}(i,e,a,s;q,\psi,\mathscr{W}) f(\psi^{(0,a')}(e,a,s))\Phi(i,e,a,ds)$$

$$= \int_{s'\in S'} f(s') T_4 \Phi(i', e', a', ds'; \Phi)$$

□

Therefore, $\mathbb{T}_4(\cdot;\Phi)$ is continuous in Φ with weak topology, we have

Lemma 27. $\mathbb{T}_4$ *has fixed points.*

Proof. Since $\mathbb{S}$ is convex and compact, M^Φ is a convex and compact set. And since $\mathbb{T}_4$ is continuous. The conditions of Schauder fixed point theorem are satisfied. $\mathbb{T}_4$ has fixed points. □

Now I construct the remaining parts of equilibrium out of $q, \psi, \mathscr{W}, \Phi$.

Proof for Proposition 1, restated in Lemma 28

Lemma 28. *For any $r > 0, \tau \in [0,1)$, and $Tr \geq 0$, $\exists \sigma_\varepsilon^*(r, \tau, Tr)$ s.t. $\forall \sigma_\varepsilon > \sigma_\varepsilon^*(r, \tau, Tr)$, a stationary equilibrium $SCE(r, \tau, Tr)$ exists.*

Proof. Given (r, τ, Tr), $w = z\dfrac{\eta - 1}{\eta}$.

Using Lemma 25, we have $\exists \sigma_\varepsilon^*$, s.t. $\forall \sigma_\varepsilon > \sigma_\varepsilon^*$, $\exists (q, \psi, \mathscr{W})$ that is a fixed point of operator $\mathbb{T}$. Then pick and fix $(q, \psi, \mathscr{W})$, using Lemma 27, we have $\exists \Phi$ that is a fixed point of operator $\mathbb{T}_4$. Now pick and fix a Φ. Construct the following (sequentially):

$$\begin{aligned}
W(i,e,a,s) &= \sigma_\varepsilon \mathscr{W}(i,e,a,s) \\
i &= \bar{r} \\
\pi &= \bar{\pi} = 1 \\
B = -[\int \sum_{a'} m^{(0,a')}(i,e,a,s;q,\psi,\mathscr{W}) q(a',e,a,s) a' \Phi(di,de,da,ds)](1+r) \\
n(e) &= v'^{-1}((1-\tau_t)w_t e) \\
N &= \sum_{e \in \mathbb{E}} \tilde{\Gamma}(e) n(e) e \\
Profit &= zN - wN \\
G &= B + Profit + \tau w N - Tr - \frac{B}{1+r} \\
Y &= zN
\end{aligned}$$

Then one verifies all the equilibrium conditions are satisfied. $\square$