Computational Methods

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About the course

Goal:

- Prepare students to solve Huggett-Aiyagari type models.

Approach:

- Step-by-step guidance to replicate papers.

For beginners:

- Review literature developed based on the framework.
- Review dynamic programming, equilibrium definitions, properties, etc.
- Review basic algorithms and numerical routines.

For advanced users:

- Learn how to fine-tune. Compare different algorithms for efficiencies.
- Introduce integration of C++ into Matlab for fast development with decent performances.

What do Huggett-Aiyagari type models do?

- Inequality:
  Aiyagari (1994), Huggett (1996), Cagetti and De Nardi (2006), Huggett et al. (2011), Kaymak and Poschke (2015), Hubmer et al. (2016)
- Consumer behavior:
  Kaplan and Violante (2010), Berger and Vavra (2015), Kaplan and Violante (2014), Telyukova (2013), Heathcote et al. (2009)
- Aggregate fluctuation:
  Krusell and Smith (1998), Heathcote (2005), Khan and Thomas (2013), Bachmann and Bai (2013), Winberry (2014)
- Taxation:
  Conesa et al. (2009), Guner et al. (2014), Badel et al. (2014)
- Monetary policy with heterogeneity:
  McKay et al. (2016), Auclert (2014), Kaplan et al. (2015), Gornemann et al. (2014), Vavra (2013)

Resources

- Books:
  Recursive Methods in Economic Dynamics - Stokey and Lucas (1989)
  Numerical Methods in Economics - Judd (1998)
  Dunamic Programming - Rust (2006)
  Numerical Recipes: The Art of Scientific Computing, Third Edition (2007)
- Others' lecture notes:
  Mark Huggett
  Dirk Krueger
  Victor Rios-Rull
  Tony Smith

Aiyagari (1994)

Decision problem:

$$\begin{aligned} v(\epsilon, k) = \max_{k'} u(c) \ &+ \beta E v[(\epsilon', k')|\epsilon] \\ s.t. \quad c + k' &\leq k(1+r) + \epsilon w l \\ k' &\geq 0, l = 1 \end{aligned}$$

Primitives: $u(\cdot)$, β, $\epsilon'|\epsilon \sim \phi(\cdot|\epsilon)$, F, δ
Equilibrium: $k'(\epsilon, k)$, Γ, K, L, r, w, s.t.

- $k'(\epsilon, k)$ solves the decision problem.
- Stationary distribution:

$$\Gamma(\mathcal{E}, \mathcal{K}) = \int 1(k'(\epsilon, k) \in \mathcal{K}, \epsilon' \in \mathcal{E}) \phi(d\epsilon'|\epsilon) \Gamma(d\epsilon, dk)$$

- Aggregation:

$$K = \int k \Gamma(dk, d\epsilon), L = \int \epsilon \Gamma(dk, d\epsilon)$$

- Price determination:

$$r = F_K(K, L) - \delta, w = F_N(K, L)$$

Aiyagari (1994), solve the decision problem

Input: w, r. Output: $k'(\epsilon, k)$
Approach: global method. Time iteration until convergence.
Algorithms:

- Grid search
- Solve optimization problem by approximating future value off-grid
- Solve the Euler equation
- Endogenous grid method based on the Euler equation

Aiyagari (1994), solve the decision problem, preliminaries

Parameterizations:

$$u(c) = c^{1-\gamma}/(1-\gamma), F = K^{\alpha} L^{1-\alpha}$$

$$\epsilon' = \rho\epsilon + \sigma(1-\rho^2)^{1/2} u, u \sim N(0,1)$$

Preliminaries:

- Discrete shock process
- Choose asset grid

Aiyagari (1994), solve the decision problem, grid search

Algorithms

- Discretize (ϵ, k) over finite grids. Choose discrete grid for k' the same as k.
- Initialize $v(\epsilon, k)$.
- Given v,
  - For each (ϵ, k), for each discrete k', compute $c(k', \epsilon, k)$
  - Compute $\tilde{v}(k', \epsilon, k) = u(c(k', \epsilon, k)) + \beta E[v(\epsilon', k')|\epsilon]$
  - Compute $v_{new}(\epsilon, k) = \max_{k'} \tilde{v}(k', \epsilon, k)$. $k'(\epsilon, k) = \arg\max$
  - Check convergence of v_{new} and v. Update v.

Disadvantages:

- Slow.
- Inaccurate. Policy function is not continuous in (w, r) or parameters.

Aiyagari (1994), solve the decision problem, approximating v function

Algorithms

- Discretize (ϵ, k).
- Initialize $v(\epsilon, k)$.
- Given v, loop over (ϵ, k)
  - For each (ϵ, k), solve problem

    $$v_{new}(\epsilon, k) = \max_{k'} u(c(k', \epsilon, k)) + \beta E[\tilde{v}(\epsilon', k')|\epsilon],$$

    where $\tilde{v}$ approximates v.
  - Check convergence of v_{new} and v. Update v.

Choice of numerical methods:

- Approximation: piece-wise (linear/spline/pchip)
- Optimization: bracketing-based method

What is Matlab slow at?

An introduction to different languages

- Compiled languages: C, C++, Fortran
  - What you write e.g. "1+1" is executed as "1+1" by machine directly
  - Need to compile after code changes. Costly to develop and debug.
- Interpreted languages: Matlab, Python
  - Everything is stored as an object internally (Matlab's mxArray, Python's PyObject)
  - When you call "1+1", the machine does the following:
    - The interpreter splits the expression as "object", "operator", "object"
    - Check the type and size of two objects and select the operator "scalar plus"
    - Read two scalars from two objects.
    - Allocate space for the result
    - Do the "1+1".
    - Wrap the result "2" into an object with its type and size.
  - When you call "A*B" for two large matrix, the same procedure goes but the bottleneck is now to do the "A*B", which is calling some external library by both C++ and Matlab. Overhead of other parts is thus less important.
  - That's why doing loops and function calls is extremely slow in Matlab.
- JIT: Julia, Matlab (partial), Java/C# (partial), Python (numba)

Aiyagari (1994), solve the decision problem, approximating v function

A vectorized implementation

Algorithms

- Discretize (ϵ, k).
- Initialize $v(\epsilon, k)$.
- Given v, vectorize over (ϵ, k)
  - For each (ϵ, k), solve problem
    $$v_{new}(\epsilon, k) = \max_{k'} u(c(k', \epsilon, k)) + \beta E[\tilde{v}(\epsilon', k')|\epsilon],$$
    where $\tilde{v}$ approximates v.
  - Check convergence of v_{new} and v. Update v.

Disadvantages:

- Write code in vectorized fashion is hard!
- Still slow!

Combining Matlab and C++: an introduction to mex

Why Matlab, not Julia or Python?

- Matlab has well-maintained libraries, version stability, free of bugs.
- Fast if used in a good way. (Built-in MKL for linear algebra and function approximation, state-of-art optimization libraries)
- Super easy integration with C++ for computation-intensive parts, while still use Matlab's other convenient utilities (debug, plot, output, etc.)

Preparations:

- Visual Studio Community 2015 on Windows. It's free for academic use.
- g++ 4.8 on Linux and Mac.
- run "mex -setup C++" in Matlab.

Aiyagari (1994), solve the decision problem, approximating v function

A C-mex implementation

Algorithms

- Discretize (ϵ, k).
- Initialize $v(\epsilon, k)$.
- Given v, loop over (ϵ, k) in C++
  - For each (ϵ, k), solve problem

    $$v_{new}(\epsilon, k) = \max_{k'} u(c(k', \epsilon, k)) + \beta E[\tilde{v}(\epsilon', k')|\epsilon],$$

    where $\tilde{v}$ approximates v.
  - Check convergence of v_{new} and v. Update v.

Advantages:

- Write code in natural ways, no need to spend extra mental cost on vectorization
- Extremely fast and parallel scalable.

Cost:

- Need to know basics about C++
- Need to be familiar with basic numerical routines (optimization, function approximation) in C++, but most are available online.

Aiyagari (1994), solve the decision problem, solve Euler equation

Decision problem:

$$v(\epsilon, k) = \max_{k'} u(c) \ + \beta E[v(\epsilon', k')|\epsilon]$$
$$s.t. \quad c + k' \leq k(1+r) + \epsilon wl$$
$$k' \geq 0, l = 1$$

FOC w.r.t. k'

$$-u'(k(1+r) + \epsilon wl - k') + \beta E[v_k(\epsilon', k')|\epsilon] + \lambda_{k' \geq 0} = 0$$

or

$$u'(k(1+r) + \epsilon wl - k') \geq \beta E[v_k(\epsilon', k')|\epsilon]$$

Equality holds iff $k' > 0$.

Aiyagari (1994), solve the decision problem, solve Euler equation

$$u'(k(1+r)+\epsilon wl-k') \geq \beta E[v_k(\epsilon',k')|\epsilon]$$

Algorithms

- Discretize (ϵ, k).
- Initialize $v_k(\epsilon, k)$.
- Given v_k, loop over (ϵ, k)
  - For each (ϵ, k), check if

    $$u'(k(1+r)+\epsilon wl-0) > \beta E[v_k(\epsilon',0)|\epsilon]$$

    if yes, then you have a corner, $k'(\epsilon, k)=0$. Else, solve k' s.t.

    $$u'(k(1+r)+\epsilon wl-k') = \beta E[\tilde{v}_k(\epsilon',k')|\epsilon]$$

    where $\tilde{v}_k$ approximates v_k.
    Since $u'(0)=\infty > \beta E[v_k(\epsilon', k(1+r)+\epsilon wl)|\epsilon]$ and
    $u'(k(1+r)+\epsilon wl-0) \leq \beta E[v_k(\epsilon',0)|\epsilon]$, we have a solution.
    Since $v_{kk}<0$ and $u''(\cdot)<0$, the solution is unique.
  - Compute $v_{k,new}$ using Envelope theorem

    $$v_{k,new}(\epsilon,k) = (1+r)u'(k(1+r)+\epsilon wl-k'(\epsilon,k))$$

  - Check convergence of v_k and update v_k.

Aiyagari (1994), solve the decision problem, endogenous grid method

$$u'(k(1+r)+\epsilon wl-k') \geq \beta E[v_k(\epsilon',k')|\epsilon]$$

Algorithm:

- Discretize (ϵ, k) over $eGrid, kGrid$.
- Initialize $v_k(\epsilon, k)$.
- Given v_k
  - For each (ϵ, k') on grids, compute
  $$\tilde{k}(\epsilon,k') = \frac{u'^{-1}(\beta E[v_k(\epsilon',k')|\epsilon]) + k' - \epsilon wl}{1+r}$$
  - Interpolate k' over $\tilde{k}$ and evaluate interpolation at $k \in kGrid$ for $k \geq \tilde{k}(\epsilon, 0)$. This gives $k'(\epsilon, k)$ for $k \geq \tilde{k}(\epsilon, 0)$.
  - We know k' is increasing in k, and therefore $k'(\epsilon,k) = 0, \forall k \leq \tilde{k}(\epsilon, 0)$.
  - Update using envelope theorem. Check convergence.

Advantages:

- No need to solve equations (avoid function calls).
- Easy to vectorize (avoid loops).

References: Carroll (2005), Barillas and Fernández-Villaverde (2007)

Aiyagari (1994), solve the decision problem, summary

Summary

- Endogenous grid method avoids repetitive function calls in optimization or root-finding. Should be the first choice whenever possible. But usually only applies to a few problems.
- Vectorized golden search deals with more general problems (1-D) and still makes computation feasible.
- For non-standard problems (any problems with states or controls go beyond 1-D) and best performances consider C-mex.

Aiyagari (1994), Monte Carlo simulations

Algorithm:

- Generate history of shocks ϵ_i^t
  - Theorem: F is a cdf, then $F^{-1}(u) \sim F$ if $u \sim Un[0,1]$
  - For discrete distribution that F is a "step" function, first generate $u \sim Un[0,1]$, then look up which "step" u falls in.
  - For Markovian process, generate ϵ_i^{t+1} base on $\phi(\cdot|\epsilon_i^t)$
- Generate initial distribution k_0.
- Simulate forward using $k'(\epsilon, k)$, approximating the policy function for k off-grid.

The choice of I and T?

- Moments of distribution become stationary.

Aiyagari (1994), solve equilibrium

Algorithm:

- Search over $r \in [r_{min}, r_{max}]$ using bracketing method:
  - Given r, solve policies, simulate stationary distribution.
  - Compute aggregate K and L from stationary distribution. Compute implied r_{new}.
- Until r and r_{new} are close enough.

Aiyagari (1994), non-stochastic simulations

Algorithm:

- Approximate distribution of states Γ with distribution $\tilde{\Gamma}$ over discrete grids (ϵ, k).
- The markovian process induced by the exogenous shock and policy function gives a transition $H : \tilde{\Gamma} \to \tilde{\Gamma}'$. Usually the approximation and transition are constructed such that the transition can be directly computed.

Example Young (2010):

- Approximate Γ as histogram over discrete grid (ϵ, k).
- Assume if $k'(\epsilon, k)$ is off-grid, it is allocated between k'_{left} and k'_{right} uniformly.

Advantages:

- No need to generate random numbers. Remove randomness of aggregate variables.
- Convenient to check convergence of distributions.

Krusell and Smith (1998): dealing with aggregate uncertainty

Decision problem:

$$\begin{aligned} v(\epsilon, k; z, \Gamma) = \max_{k'} u(c) \; &+ \beta E v[(\epsilon', k'; z', \Gamma') | \epsilon, z] \\ s.t. \quad c + k' &\leq k(1 + r(z, \Gamma)) + \epsilon w(z, \Gamma) l \\ k' &\geq 0, l = 1 \\ \Gamma' &= H(z, \Gamma) \end{aligned}$$

Primitives: $u(\cdot)$, β, $(\epsilon', z') | (\epsilon, z) \sim \phi(\cdot | \epsilon, z)$, F, δ
Equilibrium: $k'(\epsilon, k; z, \Gamma)$, $H : (z', z, \Gamma) \to \Gamma'$, $K(z, \Gamma), L(z, \Gamma), r(z, \Gamma), w(z, \Gamma)$, s.t.

- $k'(\epsilon, k; z, \Gamma)$ solves the decision problem.
- Transition rule of measure:

$$H(z', z, \Gamma)(\mathcal{K}, \mathcal{E}) = \int 1(k'(\epsilon, k; z, \Gamma) \in \mathcal{K}, \epsilon' \in \mathcal{E}) \phi(d\epsilon', dz' | \epsilon, z) \Gamma(d\epsilon, dk)$$

- Aggregation:

$$K(z, \Gamma) = \int k \Gamma(dk, d\epsilon), L(z, \Gamma) = \int \epsilon \Gamma(dk, d\epsilon)$$

- Price determination:

$$r(z, \Gamma) = F_K(z, K, L) - \delta, w(z, \Gamma) = F_N(z, K, L)$$

Krusell and Smith (1998): dealing with aggregate uncertainty

Challenge: Γ is an infinite-dimension object.
Solution by Krusell and Smith (1998): approximate Γ using finite moments.
Findings:

- First moment of k is sufficient to approximate the distribution. Why?
  - r and w are only determined by $\bar{k}$
  - Households' decision rules for saving are almost linear in k when k is large. Nonlinearality of decision rules happens when k is low, but in equilibrium the mass of people with low k is small.
- Transtion rule for distribution $H(z', z, \Gamma)$ can be well approximated by

$$\log(\bar{k}') = \alpha_0(z) + \alpha_1(z)\log(\bar{k})$$

Krusell and Smith (1998): dealing with aggregate uncertainty

Decision problem:

$$
\begin{aligned}
v(\epsilon, k; z, \bar{k}) = \max_{k'} u(c) \; &+ \beta E v[(\epsilon', k'; z', \bar{k}')|\epsilon, z] \\
s.t. \quad c + k' &\leq k(1 + r(z, \bar{k})) + \epsilon w(z, \bar{k}) l \\
k' &\geq 0, l = 1 \\
\log(\bar{k}') &= \alpha_0(z) + \alpha_1(z) \log(\bar{k})
\end{aligned}
$$

Primitives: $u(\cdot)$, β, $(\epsilon', z')|(\epsilon, z) \sim \phi(\cdot|\epsilon, z)$, F, δ
Equilibrium: $k'(\epsilon, k; z, \bar{k})$, $\alpha_0(z), \alpha_1(z)$, $K(z, \bar{k}), L(z, \bar{k}), r(z, \bar{k}), w(z, \bar{k})$, s.t.

- $k'(\epsilon, k; z, \bar{k})$ solves the decision problem.
- Transition rule of moment is consistent with the transition of distribution.
- Aggregation:

$$\bar{k} = \int k\Gamma(dk, d\epsilon), L(z, \bar{k}) = \int \epsilon\Gamma(dk, d\epsilon)$$

- Price determination:

$$r(z, \bar{k}) = F_K(z, \bar{k}, L) - \delta, w(z, \bar{k}) = F_N(z, \bar{k}, L)$$

Krusell and Smith (1998): solve the decision problem

$$
\begin{aligned}
v(\epsilon, k; z, \bar{k}) = \max_{k'} u(c) \ + \beta E v[(\epsilon', k'; z', \bar{k}')|\epsilon, z] \\
s.t. \quad c + k' \leq k(1 + r(z, \bar{k})) + \epsilon w(z, \bar{k}) l \\
k' \geq 0, l = 1 \\
\log(\bar{k}') = \alpha_0(z) + \alpha_1(z) \log(\bar{k})
\end{aligned}
$$

Challenges: how to interp over two dimensions $(k', \bar{k}')$.
Algorithm:

- Given $(z, \bar{k})$, compute $\bar{k}'$ using the transition rule.
- Interpolate over $\bar{k}'$, get $(E\tilde{v})_{z,\bar{k}',\epsilon}(k')$. That is, when we solve individual's problem varying k', $(z, \bar{k}')$ are always fixed.
- Solve individual's optimization problem.

Krusell and Smith (1998): simulations and aggregation

Algorithm:

- Simulate k_t forward using policy function $k'(\epsilon, k; z, \bar{k})$. Construct $\bar{k}_t$ along the way.
- Run regressions for samples $z_t = B$ and $z_t = G$.

$$\log(\bar{k}_{t+1}) = \alpha_0(z_t) + \alpha_1(z_t)\log(\bar{k}_t)$$

- Update coefficients. Resolve the decision problem, simulation, regression.

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